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Nonparametric Inference on Multivariate Versions of Blomqvist's Beta and Related Measures of Tail Dependence

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SUMMARY

We consider nonparametric estimation of multivariate versions of Blomqvist's beta, also known as the medial correlation coefficient. For a two-dimensional population, the sample version of Blomqvist's beta describes the proportion of data which fall into the first or third quadrant of a two-way contingency table with cutting points being the sample medians. Asymptotic normality and strong consistency of the estimators are established by means of the empirical copula process, imposing weak conditions on the copula. Though the asymptotic variance takes a complicated form, we are able to derive explicit formulas for large families of copulas. For the copulas of elliptically contoured distributions we obtain a variance stabilizing transformation which is similar to Fisher's z-transformation. This allows for an explicit construction of asymptotic confidence bands used for hypothesis testing and eases the analysis of asymptotic efficiency. The computational complexity of estimating Blomqvist's beta corresponds to the sample size n, which is lower than the complexity of most competing dependence measures.

Keywords: Blomqvist's beta, copula, tail dependence, asymptotic normality, empirical copula, asymptotic efficiency.

AMS 2000 subject class.: Primary 62H20, 62G20, Secondary 62G30, 62H10.

1 Introduction

Suppose $(x_1, y_1), \ldots, (x_n, y_n)$, $n \in \mathbb{N}$, is a sample of a two-dimensional population with joint distribution function F and continuous marginal distribution functions. Consider the corresponding two-way contingency table with cutting points being the sample medians of the respective margins. In that case, the cell counts in the contingency table do not follow a multinomial distribution. Let n_1 denote the number of sample data which belong to the first or third quadrant of the table and n_2 denote the number belonging to the second or fourth quadrant. If the sample size is odd, we

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adjust this definition appropriately. Blomqvist (1950) suggested the following simple dependence measure, which is based on the above contingency table:

$$\frac{n_1 - n_2}{n_1 + n_2} = \frac{2n_1}{n_1 + n_2} - 1 =: \hat{\beta}_n. \tag{1}$$

This measure is commonly referred to as Blomqvist's β or the medial correlation coefficient. For a pair of continuous random variables X and Y, the population version of $\hat{\beta}_n$, as given in (1), takes the form

$$\beta := P\{(X - \tilde{x})(Y - \tilde{y}) > 0\} - P\{(X - \tilde{x})(Y - \tilde{y}) < 0\}, \tag{2}$$

where $\tilde{x}$ and $\tilde{y}$, respectively, denote the median of X and Y. Several generalizations of Blomqvist's β to $d > 2$ dimensions have previously been developed in the literature. We mention Joe (1990), Nelsen (2002), Taskinen, Oja, and Randles (2005), and Úbeda-Flores (2005). Most of them are based on the notion of a copula which we shortly recall in the next section.

As a measure of multivariate dependence, Blomqvist's β has some advantages over competing measures such as Spearman's ρ or Kendall's τ. First, Blomqvist's β can explicitly be derived whenever the copula is of explicit form, what is often not possible for the previously mentioned dependence measures, see Joe (1997), Chapter 5. Even for the copulas of elliptically contoured distributions - most of them are given implicitly - we are able to derive explicit formulas for Blomqvist's β. This family of copulas is frequently encountered in financial engineering, see Cherubini et al. (2004) for an overview. Another advantage of Blomqvist's β is the low computational complexity of its estimation. Whereas most implementations of Spearman's ρ and Kendall's τ have a complexity of n^2, the estimation of Blomqvist's β has a complexity of n. This holds because only the median of the univariate margins needs to be determined (Schöning 1997, p.69). Thus, Blomqvist's β represents a fast alternative of estimating copula parameters via the 'method of moments'. Further, although the asymptotic theory of (multivariate versions of) Spearman's ρ and Kendall's τ has already been developed - see e.g. Rüschendorf (1976), Ruymgaart and van Zuijlen (1978), Gänßler and Stute (1987), Joe (1990), Stepanova (2003), Schmid and Schmidt (2006a) - and compact expressions for the related asymptotic variances are known, they can only be evaluated in very special cases since they involve multiple integration. On the contrary, the asymptotic variance of the estimator $\hat{\beta}_n$ is much simpler and can be explicitly derived for large families of copulas as we will see in Section 5. For the copulas of elliptically contoured distributions we even obtain a variance stabilizing transformation which is similar to Fisher's z-transformation. This opens the possibility to use standardized $\hat{\beta}_n$ not only as a test statistics for testing independence, but also for testing general dependence structures.

The aim of this paper is twofold. First, we generalize Blomqvist's β in order to obtain a new class of multivariate dependence measures which measure the amount of dependence in the tail region of a distribution of d random variables $(X_1, \ldots, X_d)$. A related class of (conditional) dependence measures based on Spearman's ρ has been developed in Schmid and Schmidt (2006b). Both families of dependence measures are increasing with respect to the multivariate concordance ordering and are determined by the copula only. Thus both are invariant with respect to the distributions of the margins X_i. In particular in financial engineering, these types of (conditional) dependence measures are frequently used in order to investigate the effects of contagion between financial markets, see Campbell, Koedijk, and Kofman (2002) or Forbes and Rigobon (2002). Our second aim is the nonparametric estimation of Blomqvist's β and its multivariate versions. The estimators we utilize are expressed by the empirical copula process, see Rüschendorf (1976), Deheuvels (1979), Fermanian et al. (2004) or Tsukahara (2005) and related techniques are used in order to establish asymptotic normality and strong consistency. As a byproduct we can show that for $d = 2$ the asymptotic distribution derived in Blomqvist's original paper arises if the marginal distributions of the underlying population are known. This is incorrect for $\hat{\beta}_n$ in (1), which is also reported in Borkowf et al. (1997).

The paper is organized as follows. Section 2 introduces some notation and gives a short account on copulas. In Section 3 we define the multivariate versions of Blomqvist's β (in short: Blomqvist's multivariate β) and examine some of their analytical properties. Section 4 deals with the nonparametric estimation under the assumption of known or unknown marginal distributions. A nonparametric bootstrap for estimating the asymptotic variance is stated at the end of the section. Various families of copulas are examined in Section 5. Explicit formulas of the asymptotic variance are calculated for a number of Archimedean and elliptical copulas. Amongst others, it is shown that a subclass of Blomqvist's multivariate β is invariant with respect to the characteristic generator of d-dimensional elliptically contoured distributions. A simulation study investigates the performance of the nonparametric bootstrap procedure. Section 6 establishes a relationship between the coefficient of (extremal) tail dependence λ_L - which is based on Blomqvist's multivariate β - and the family of elliptically contoured distributions. In Section 7 we address asymptotic efficiency. Relative efficiency and Pitman efficiency of Blomqvist's multivariate β with respect to multivariate versions of Spearman's ρ are considered.

2 Notation and definitions

Let $X_1, X_2, \ldots, X_d$, $d \geq 2$, be random variables with joint distribution function

$$F(\mathbf{x}) = P(X_1 \leq x_1, \ldots, X_d \leq x_d), \quad \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

and marginal distribution functions $F_i(x_i) = P(X_i \leq x_i)$ for $x_i \in \mathbb{R}$ and $i = 1, \ldots, d$. Throughout the paper we write bold letters for vectors, e.g., $\mathbf{x} = (x_1, \ldots, x_d)$ is a d-dimensional vector. Inequalities $\mathbf{x} \leq \mathbf{y}$ are understood componentwise, i.e, $x_i \leq y_i$ for all $i = 1, \ldots, d$. Let $\ell^\infty(T)$ denote the space of all uniformly bounded real-valued functions on some set T. The indicator function on a set A is denoted by $\mathbf{1}_A$. If not stated otherwise, we assume that the F_i are continuous functions. Thus, according to Sklar's theorem (Sklar 1959), there exists a unique copula $C : [0,1]^d \to [0,1]$ such that

$$F(\mathbf{x}) = C(F_1(x_1), \ldots, F_d(x_d)), \quad \text{for all } \mathbf{x} \in \mathbb{R}^d.$$

The copula C is the joint distribution function of the random variables $U_i = F_i(X_i)$, $i = 1, \ldots, d$. Moreover, $C(\mathbf{u}) = P(\mathbf{U} \leq \mathbf{u}) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))$ for all $\mathbf{u} \in [0,1]^d$. The generalized inverse function G^{-1} is defined via $G^{-1}(u) := \inf\{x \in \mathbb{R} \cup \{\infty\} \mid G(x) \geq u\}$ for all $u \in (0,1]$ and $G^{-1}(0) := \sup\{x \in \mathbb{R} \cup \{-\infty\} \mid G(x) = 0\}$. We will need the survival function $\bar{C}(\mathbf{u}) = P(\mathbf{U} > \mathbf{u})$. It is well known that every copula C is bounded in the following sense:

$$\begin{aligned} W(\mathbf{u}) &:= \max\{u_1 + \ldots + u_d - (d-1), 0\} \\ &\leq C(\mathbf{u}) \leq \min\{u_1, \ldots, u_d\} =: M(\mathbf{u}) \quad \text{for all } \mathbf{u} \in [0,1]^d, \end{aligned}$$

where W and M are called the lower and upper *Fréchet-Hoeffding bounds*, respectively. The upper bound M is a copula itself and is also known as the comonotonic copula. It represents the copula of $X_1, \ldots, X_d$ if $F_1(X_1) = \cdots = F_d(X_d)$ with probability one, i.e., if there exists an almost surely strictly increasing functional relationship between X_i and X_j. By contrast, the lower bound W is a copula only for dimension $d = 2$. Another important copula is the independence copula

$$\Pi(\mathbf{u}) := \prod_{i=1}^d u_i, \quad \mathbf{u} \in [0,1]^d,$$

which describes the dependence structure of stochastically independent random variables $X_1, \ldots, X_d$. A detailed treatment of copulas is given in Joe (1997) and Nelsen (2006).

It is easy to see that Blomqvist's β, as given in (2), can be expressed via the copula C of X and Y, namely

$$\beta = 2P\{(X - \tilde{x})(Y - \tilde{y}) > 0\} - 1 = 4C(\mathbf{1/2}) - 1 = \frac{C(\mathbf{1/2}) - \Pi(\mathbf{1/2}) + \bar{C}(\mathbf{1/2}) - \bar{\Pi}(\mathbf{1/2})}{M(\mathbf{1/2}) - \Pi(\mathbf{1/2}) + \bar{M}(\mathbf{1/2}) - \bar{\Pi}(\mathbf{1/2})} \quad (3)$$

with notation $\mathbf{1/2} = (1/2, 1/2)$. Thus, β can be interpreted as a normalized distance between the copula C and the independence copula Π. We also mention that β can be written as the Bravais-Pearson correlation coefficient of $sign(X - \tilde{x})$ and $sign(Y - \tilde{y})$, i.e. $\beta = E\{sign(X - \tilde{x}) \cdot sign(Y - \tilde{y})\}$. The right-hand formula in (3) will motivate the following multivariate versions of Blomqvist's β.

3 Multivariate Versions of Blomqvist's β

Suppose C is the copula of d random variables $X_1, \ldots, X_d$. A first multivariate version of Blomqvist's β is defined by

$$\beta: \quad = \quad \frac{C(\mathbf{1/2}) - \Pi(\mathbf{1/2}) + \bar{C}(\mathbf{1/2}) - \bar{\Pi}(\mathbf{1/2})}{M(\mathbf{1/2}) - \Pi(\mathbf{1/2}) + \bar{M}(\mathbf{1/2}) - \bar{\Pi}(\mathbf{1/2})} = h_d\left\{C(\mathbf{1/2}) + \bar{C}(\mathbf{1/2}) - 2^{1-d}\right\}, \qquad (4)$$

where $h_d := 2^{d-1}/(2^{d-1}-1)$ and $\mathbf{1/2} := (1/2, \ldots, 1/2)$. Obviously $M(\mathbf{1/2}) = 1/2$ and $\Pi(\mathbf{1/2}) = 2^{-d}$. This multivariate version of Blomqvist's β coincides with the bivariate version, as given in formula (3) for dimension $d = 2$. Note that $C\left(\frac{1}{2}, \frac{1}{2}\right) = \bar{C}\left(\frac{1}{2}, \frac{1}{2}\right)$ holds for any bivariate copula C. In general, $C(\mathbf{1/2}) \neq \bar{C}(\mathbf{1/2})$ for $d \geq 3$.

The next generalization of Blomqvist's β plays the central role in our analysis. The generalization is such that emphasis is put on the tail regions of the copula which determine the degree of large co-movements between the marginal random variables. For example in financial markets, large co-movements of (negative) asset-returns occur often during market crash or stress situations, cf. e.g. Karolyi and Stulz (1996), Longin and Solnik (2001), or Forbes and Rigobon (2002). Financial engineers are particularly interested in the precise modelling and measuring of this kind of dependence since it principally determines the distribution of large losses in a portfolio, see Ong (1999). We do not fix the point $\mathbf{1/2}$ (or the median) in formula (4) anymore, but allow for arbitrary cutting points; we define

$$\beta(\mathbf{u}, \mathbf{v}) := h_d(\mathbf{u}, \mathbf{v})\left[\left\{C(\mathbf{u}) + \bar{C}(\mathbf{v})\right\} - g_d(\mathbf{u}, \mathbf{v})\right] \quad \text{for } (\mathbf{u}, \mathbf{v}) \in D, \qquad (5)$$

with norming constants $h_d(\mathbf{u}, \mathbf{v}) = \{\min(u_1, \ldots, u_d) + \min(1 - v_1, \ldots, 1 - v_d) - \prod_{i=1}^d u_i - \prod_{i=1}^d (1 - v_i)\}^{-1}$ and $g_d(\mathbf{u}, \mathbf{v}) = \prod_{i=1}^d u_i + \prod_{i=1}^d (1 - v_i)$. The domain of the cutting points is $D := \{(\mathbf{u}, \mathbf{v}) \in [0,1]^{2d} \mid \mathbf{u} \leq \mathbf{v}, \mathbf{u} > \mathbf{0} \text{ or } \mathbf{v} < \mathbf{1}\}$. In the following we distinguish between three important cases:

i) $\beta = \beta(\mathbf{1/2}, \mathbf{1/2})$ which measures the 'overall' dependence,

ii) $\beta(\mathbf{u}, \mathbf{v})$ with $\mathbf{u} < \mathbf{1/2} < \mathbf{v}$ which measures dependence in the tail region,

iii) $\lim_{p \downarrow 0} \beta\{(p, \ldots, p), \mathbf{1}\}$ which measures (lower) extremal dependence.

Case iii) is a multivariate version of the so-called tail-dependence coefficient which measures bivariate tail dependence and plays a role in extreme value theory, see Section 6. Though from a theoretical point of view measures of extremal dependence are interesting, they have there limitations in application due to the finite number of observations. Usually the statistician fixes a cutting point which splits the extremal observations from the non-extremal observations. The statistical inference is then based on extreme value techniques whose performance is sensitive to the choice of the cutting point, see e.g. Frahm et al. (2005). The measure defined in case ii) also measures dependence in the tail of a distribution utilizing a cutting point. However, in the present paper, the statistical inference is elaborated without using extreme value techniques and its applicability is shown.

Alternative representations of β are developed in a pre-version of this paper which is available on the authors' webpage. Further multivariate versions of Blomqvist's β - such as an average pairwise version - are given, for example, in Joe (1990) or Úbeda-Flores (2005).

Multivariate concordance. An order of multivariate concordance $C \prec_c C^*$ - in the sense of Joe (1990) - between two copulas C and C^* holds if

$$C(\mathbf{u}) \leq C^*(\mathbf{u}) \text{ and } \bar{C}(\mathbf{u}) \leq \bar{C}^*(\mathbf{u}) \qquad \text{for every } \mathbf{u} \in [0,1]^d.$$

Scarsini (1984) proposed a set of axioms for a measure of bivariate concordance, see also Nelsen (2006), Definition 5.1.7. Various sets of axioms for a measure of multivariate concordance have recently been proposed by Dolati and Úbeda-Flores (2006) and Taylor (2006). The following axioms are fulfilled by $\beta(\mathbf{u}, \mathbf{v})$:

i) $\beta(\mathbf{u}, \mathbf{v})$ is well-defined for every d-dimensional random vector $\mathbf{X}$ ($d \geq 2$) with continuous margins X_i and it is solely determined by its (unique) copula C;

ii) $-1 \leq \beta(\mathbf{u}, \mathbf{v}) \leq 1$ and $\beta(\mathbf{u}, \mathbf{v}) = 1$ if $C = M$;

iii) $\beta_{\mathbf{X}}(\mathbf{u}, \mathbf{v}) = \beta_{\pi(\mathbf{X})}(\mathbf{u}, \mathbf{v})$ for any permutation π of the components of $\mathbf{X}$ if $\mathbf{u} = (u, \ldots, u)$ and $\mathbf{v} = (v, \ldots, v)$, i.e., where $\mathbf{u}, \mathbf{v}$ have equal coordinates.

iv) if $\mathbf{X}$ has stochastically independent components, then $\beta(\mathbf{u}, \mathbf{v}) = 0$;

v) if C and C^* are copulas such that $C \prec_c C^*$, then $\beta_C(\mathbf{u}, \mathbf{v}) \leq \beta_{C^*}(\mathbf{u}, \mathbf{v})$;

vi) if $(\mathbf{X}_n)_{n \in \mathbb{N}}$ is a sequence of random vectors with continuous margins $X_{i,n}$ and copulas C_n, and if $\lim_{n \to \infty} C_n(\mathbf{u}) = C(\mathbf{u})$ for all $\mathbf{u} \in [0, 1]^d$ and some copula C, then $\lim_{n \to \infty} \beta_{C_n}(\mathbf{u}, \mathbf{v}) = \beta_C(\mathbf{u}, \mathbf{v})$.

4 Estimation

This section deals with the estimation of the multivariate versions of Blomqvist's β. We consider estimators under the assumption of completely known and completely unknown marginal distributions. For each estimator, asymptotic normality is established and the differences of the asymptotic variances are elaborated.

4.1 Estimation under known margins

Let $\mathbf{X}$ be a d-dimensional random vector with distribution function F, continuous marginal distribution functions F_i, and copula C. Suppose $(\mathbf{X}_j)_{j=1,\ldots,n}$ is a random sample generated from $\mathbf{X}$. In the present section we assume that the univariate marginal distribution functions F_i of F are continuous and known. We set $U_{ij} := F_i(X_{ij}),\ i = 1, \ldots, d,\ j = 1, \ldots, n$. Thus, the random vectors $\mathbf{U}_j = (U_{1j}, \ldots, U_{dj}),\ j = 1, \ldots, n$, are distributed according to the copula C. The assumption of known marginal distributions is dropped in the next section. We make that assumption at this point in order to show that the asymptotic distribution calculated by Blomqvist (1950), Chapter 4, arises in that special case. Blomqvist considered the case $\beta = \beta(\mathbf{1/2}, \mathbf{1/2})$, thus, we first confine ourselves to this dependence measure. The utilized estimator is of the form

$$\hat{\beta}_n^\star = h_d \left\{ \frac{1}{n} \sum_{j=1}^{n} \left(\prod_{i=1}^{d} \mathbf{1}_{\{U_{ij} \leq \frac{1}{2}\}} + \prod_{i=1}^{d} \mathbf{1}_{\{U_{ij} > \frac{1}{2}\}} \right) - 2^{1-d} \right\}. \tag{6}$$

The next proposition states the asymptotic normality of the estimator. The *proof* follows by an application of the central limit theorem.

Proposition 1 *Suppose F is a d-dimensional distribution function with known continuous marginal distribution functions and copula C. Let $\hat{\beta}_n^\star$ be the estimator defined in (6). Then*

$$\sqrt{n}(\hat{\beta}_n^\star - \beta) \xrightarrow{d} Z \quad \textit{with} \quad Z \sim N(0, \sigma_\star^2)$$

and $\sigma_\star^2 = 2^{2d-2}[C(\mathbf{1/2}) + \bar{C}(\mathbf{1/2}) - \{C(\mathbf{1/2}) + \bar{C}(\mathbf{1/2})\}^2]/(2^{d-1} - 1)^2$.

Example. For the d-dimensional independence copula Π we have $C(\mathbf{1/2}) = \bar{C}(\mathbf{1/2}) = 2^{-d}$ and thus, $\sigma_\star^2 = 1/(2^{d-1} - 1)$.

Remark. For dimension $d = 2$, β can always be represented by $\beta = 4C(\mathbf{1/2}) - 1$, cf. formula (3), which motivates the consideration of the following estimator for β :

$$4 \sum_{j=1}^{n} \mathbf{1}_{\{U_{1j} \leq 1/2 \text{ and } U_{2j} \leq 1/2\}} - 1. \tag{7}$$

This estimator possesses an asymptotic variance of $4[C(\mathbf{1/2}) - \{C(\mathbf{1/2})\}^2]$ which is larger than $\sigma_\star^2$ for any copula C. For example, if C is the independence copula Π, the asymptotic variance of the latter estimator is three times the asymptotic variance of $\hat{\beta}_n^\star$. This is intuitive since the estimator in (7) retrieves less information from the data than $\hat{\beta}_n^\star$.

Blomqvist (1950) claims that $\sqrt{n}(\hat{\beta}_n - \beta)$, as defined in formulas (1) and (2), is asymptotically normal with variance $1-q^2$, where $q = 4\bar{C}(\mathbf{1/2}) - 1 = 4C(\mathbf{1/2}) - 1$. The asymptotic variance $\sigma_\star^2$ for dimension $d=2$, as given in Proposition 1, coincides with $1 - q^2 = 1 - \{4C(\mathbf{1/2}) - 1\}^2 = 4[2C(\mathbf{1/2}) - \{2C(\mathbf{1/2})\}^2] = \sigma_\star^2$. This implies that the asymptotic variance derived in the above reference equals the asymptotic variance of $\sqrt{n}(\hat{\beta}_n^\star - \beta)$. In other words, the asymptotic variance is valid if the marginal distributions or the corresponding medians are known. By contrast, Corollary 3 later shows that $\sqrt{n}(\hat{\beta}_n - \beta)$ does not possess this asymptotic variance in general. In particular, Blomqvist (1950) utilized an incorrect change of limits in his formula (7). For the Ali-Mikhail-Haq copula $C(u_1, u_2) = u_1 u_2 / \{1 - \theta(1-u_1)(1-u_2)\}$, $\theta \in [-1, 1)$, Figure 1 illustrates the differences between these asymptotic variances over θ. Only for $\theta = 0$ - which corresponds to the independence copula Π - the variances coincide.

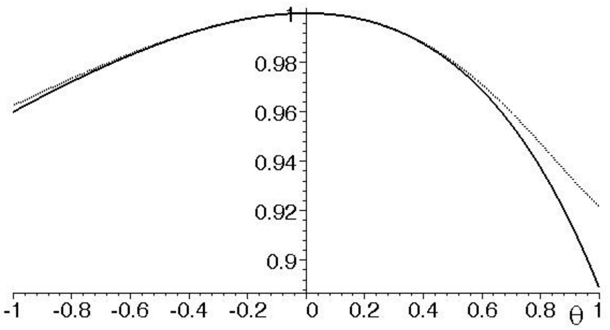

Figure 1: Asymptotic variance of Blomqvist's $\hat{\beta}_n$ (dotted line) and $\hat{\beta}_n^\star$ (solid line) for the Ali-Mikhail-Haq copula depending on the parameter θ.

4.2 Estimation under unknown margins

We drop the assumption of known marginal distributions and discuss nonparametric estimators for the versions of Blomqvist's β defined in (5). More precisely, we consider a random sample $(\mathbf{X}_j)_{j=1,\ldots,n}$ from a d-dimensional random vector $\mathbf{X}$ with joint distribution function F and copula C which are completely unknown. The marginal distribution functions F_i are estimated by their empirical counterparts

$$\hat{F}_{i,n}(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{X_{ij} \le x\}}, \quad \text{for } i = 1, \ldots, d \text{ and } x \in \bar{\mathbb{R}}.$$

Further, set $\hat{U}_{ij,n} := \hat{F}_{i,n}(X_{ij})$ for $i = 1, \ldots, d$, $j = 1, \ldots, n$, and $\hat{\mathbf{U}}_{j,n} = (\hat{U}_{1j,n}, \ldots, \hat{U}_{dj,n})$. Note that

$$\hat{U}_{ij,n} = \frac{1}{n}(\text{rank of } \ X_{ij} \ \text{ in } \ X_{i1}, \ldots, X_{in}). \tag{8}$$

The estimation of β will therefore be based on ranks and not on the observations themselves. The copula C and the survival function $\bar{C}$ are estimated using the empirical copula C_n and the empirical survival function $\bar{C}_n$ which are defined by

$$C_n(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d} \mathbf{1}_{\{\hat{U}_{ij,n} \le u_i\}} \text{ and } \bar{C}_n(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d} \mathbf{1}_{\{\hat{U}_{ij,n} > u_i\}} \text{ for } \mathbf{u} = (u_1, \ldots, u_d)' \in [0,1]^d.$$

The natural estimator for $\beta(\mathbf{u},\mathbf{v})$, as stated in formula (5), is then given by

$$\hat{\beta}_n(\mathbf{u},\mathbf{v}) = h_d(\mathbf{u},\mathbf{v}) \left[\left\{ C_n\left(\mathbf{u}\right) + \bar{C}_n\left(\mathbf{v}\right) \right\} - g_d(\mathbf{u},\mathbf{v}) \right]. \tag{9}$$

with norming constants $h_d(\mathbf{u},\mathbf{v})$ and $g_d(\mathbf{u},\mathbf{v})$ defined in (5). The asymptotic distribution of $\sqrt{n}\{\hat{\beta}_n(\mathbf{u},\mathbf{v}) - \beta(\mathbf{u},\mathbf{v})\}$ is established next.

Theorem 2 *Let F be a continuous d-dimensional distribution function with copula C. Set $D_\varepsilon := \{(\mathbf{u},\mathbf{v}) \in [0,1]^{2d} \mid \mathbf{u} \leq \mathbf{v},\ \mathbf{u} \geq \varepsilon \text{ or } \mathbf{v} \leq \mathbf{1} - \varepsilon\}$ for arbitrary but fixed $\varepsilon > \mathbf{0}$. Under the additional assumption that the i-th partial derivatives D_iC and $D_i\bar{C}$ exist and are continuous:*

$$\sqrt{n}\left\{\hat{\beta}_n(\mathbf{u},\mathbf{v}) - \beta(\mathbf{u},\mathbf{v})\right\} \xrightarrow{w} \mathbb{G}(\mathbf{u},\mathbf{v}),$$

where $\mathbb{G}(\mathbf{u},\mathbf{v})$ is a centered tight continuous Gaussian process. Weak convergence takes place in $\ell^\infty(D_\varepsilon)$ and

$$\mathbb{G}(\mathbf{u},\mathbf{v}) = h_d(\mathbf{u},\mathbf{v}) \left[\mathbb{B}_C\left(\mathbf{u}\right) + \mathbb{B}_{\bar{C}}\left(\mathbf{v}\right) - \sum_{i=1}^{d} \left\{ D_iC\left(\mathbf{u}\right)\mathbb{B}_C\left(\mathbf{u}^{(i)}\right) + D_i\bar{C}\left(\mathbf{v}\right)\mathbb{B}_C\left(\mathbf{v}^{(i)}\right) \right\} \right] \tag{10}$$

with $\mathbf{u}^{(i)} := (1,...,1,u_i,1,...,1)$. Here, $\mathbb{B}_C$ and $\mathbb{B}_{\bar{C}}$ are centered tight Gaussian processes on $[0,1]^d$ with covariance functions $E\{\mathbb{B}_C(\mathbf{u}_1)\mathbb{B}_C(\mathbf{u}_2)\} = C(\mathbf{u}_1 \wedge \mathbf{u}_2) - C(\mathbf{u}_1)C(\mathbf{u}_2)$ and $E\{\mathbb{B}_{\bar{C}}(\mathbf{u}_1)\mathbb{B}_{\bar{C}}(\mathbf{u}_2)\} = \bar{C}(\mathbf{u}_1 \vee \mathbf{u}_2) - \bar{C}(\mathbf{u}_1)\bar{C}(\mathbf{u}_2)$. Moreover, $\mathbb{B}_C\left(\mathbf{u}\right)$ and $\mathbb{B}_{\bar{C}}\left(\mathbf{v}\right)$ are jointly normally distributed with covariance

$$E\{\mathbb{B}_C(\mathbf{u})\mathbb{B}_{\bar{C}}(\mathbf{v})\} = -C(\mathbf{u})\bar{C}(\mathbf{v}). \tag{11}$$

Proof. Let $(\mathbf{u},\mathbf{v}) \in D_\varepsilon,\ \varepsilon > \mathbf{0}$, and $\bar{F}(\mathbf{x}) = P(\mathbf{X} > \mathbf{x})$. The following holds:

$$\begin{aligned}
&\sqrt{n}(\hat{\beta}_n(\mathbf{u},\mathbf{v}) - \beta(\mathbf{u},\mathbf{v}))/h_d(\mathbf{u},\mathbf{v}) + O(1/\sqrt{n}) \\
&= \quad \sqrt{n}\Big[\hat{F}_n\Big\{\hat{F}_{1,n}^{-1}(u_1),\ldots,\hat{F}_{d,n}^{-1}(u_d)\Big\} + \hat{\bar{F}}_n\Big\{\hat{F}_{1,n}^{-1}(v_1),\ldots,\hat{F}_{d,n}^{-1}(v_d)\Big\} - \{C(\mathbf{u}) + \bar{C}(\mathbf{v})\}\Big] = (*),
\end{aligned}$$

with empirical joint distribution and survival function

$$\hat{F}_n(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{X_{ij} \leq x_i\}} \quad \text{and} \quad \hat{\bar{F}}_n(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{X_{ij} > x_i\}}. \tag{12}$$

It is well known (see e.g. Example 2.10.4 in Van der Vaart and Wellner (1996)) that the empirical processes $H_n = \sqrt{n}(\hat{F}_n - F)$ and $\bar{H}_n = \sqrt{n}(\hat{\bar{F}}_n - \bar{F})$ converge weakly in $\ell^\infty([-\infty,\infty]^d)$ to centered tight continuous Gaussian processes. Similarly, the empirical process $H_n(\mathbf{x}) + \bar{H}_n(\mathbf{y})$ converges weakly in $\ell^\infty([-\infty,\infty]^{2d})$ to a centered tight Gaussian process. The verification of this is standard; for example, marginal convergence is proven via a multivariate version of the Lindeberg-Feller theorem for triangular arrays, cf. Araujo and Giné (1980), p.41.

We have

$$
\begin{aligned}
(*) \quad = \quad & H_n\{F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)\} + \bar{H}_n\{F_1^{-1}(v_1),\ldots,F_d^{-1}(v_d)\} && (13)\\
& +\sqrt{n}\Big[F\{\hat{F}_{1,n}^{-1}(u_1),\ldots,\hat{F}_{d,n}^{-1}(u_d)\} - C(\mathbf{u})\Big] \\
& +\sqrt{n}\Big[\bar{F}\{\hat{F}_{1,n}^{-1}(v_1),\ldots,\hat{F}_{d,n}^{-1}(v_d)\} - \bar{C}(\mathbf{v})\Big] \\
& +\sum_{i=1}^{d}\Big[H_n\{F_1^{-1}(u_1),\ldots,F_{i-1}^{-1}(u_{i-1}),\hat{F}_{i,n}^{-1}(u_i),\ldots,\hat{F}_{d,n}^{-1}(u_d)\} && (14)\\
& \qquad -H_n\{F_1^{-1}(u_1),\ldots,F_i^{-1}(u_i),\hat{F}_{i+1,n}^{-1}(u_{i+1}),\ldots,\hat{F}_{d,n}^{-1}(u_d)\}\Big] \\
& +\sum_{i=1}^{d}\Big[\bar{H}_n\{F_1^{-1}(v_1),\ldots,F_{i-1}^{-1}(v_{i-1}),\hat{F}_{i,n}^{-1}(v_i),\ldots,\hat{F}_{d,n}^{-1}(v_d)\} && (15)\\
& \qquad -\bar{H}_n\{F_1^{-1}(v_1),\ldots,F_i^{-1}(v_i),\hat{F}_{i+1,n}^{-1}(v_{i+1}),\ldots,\hat{F}_{d,n}^{-1}(v_d)\}\Big].
\end{aligned}
$$

The sums in formulas (14) and (15) converge to zero in probability due to the weak convergence of H_n and $\bar{H}_n$ in $\ell^\infty([-\infty,\infty]^d)$; utilize tightness of H_n and $\bar{H}_n$. The expression in formula (13) converges in distribution to $\mathbb{B}_C(\mathbf{u}) + \mathbb{B}_{\bar{C}}(\mathbf{v})$, where $\mathbb{B}_C$ and $\mathbb{B}_{\bar{C}}$ are d-dimensional Brownian sheets with the following covariance functions on $[0,1]^d$:

$$E\{\mathbb{B}_C(\mathbf{u}_1)\mathbb{B}_C(\mathbf{u}_2)\} = C(\mathbf{u}_1\wedge\mathbf{u}_2)-C(\mathbf{u}_1)C(\mathbf{u}_2) \text{ and } E\{\mathbb{B}_{\bar{C}}(\mathbf{u}_1)\mathbb{B}_{\bar{C}}(\mathbf{u}_2)\} = \bar{C}(\mathbf{u}_1\vee\mathbf{u}_2)-\bar{C}(\mathbf{u}_1)\bar{C}(\mathbf{u}_2).$$

The covariance structure between $\mathbb{B}_C(\mathbf{u})$ and $\mathbb{B}_{\bar{C}}(\mathbf{v})$, which are jointly normally distributed, is of the form (11). Moreover,

$$
\begin{aligned}
& \sqrt{n}\Big[F\{\hat{F}_{1,n}^{-1}(u_1),\ldots,\hat{F}_{d,n}^{-1}(u_d)\} - C(\mathbf{u})\Big] + \sqrt{n}\Big[\bar{F}\{\hat{F}_{1,n}^{-1}(v_1),\ldots,\hat{F}_{d,n}^{-1}(v_d)\} - \bar{C}(\mathbf{v})\Big] \\
& \xrightarrow{w} -\sum_{i=1}^{d}\Big\{D_iC\left(\mathbf{u}\right)\mathbb{B}_C\left(\mathbf{u}^{(i)}\right) + D_i\bar{C}\left(\mathbf{v}\right)\mathbb{B}_C\left(\mathbf{v}^{(i)}\right)\Big\}
\end{aligned}
$$

due to the prerequisites and the Bahadur representation of the uniform empirical process. An application of the functional Delta method (Van der Vaart and Wellner 1996, p.374) to the map $\phi(F,\bar{F})(\mathbf{u},\mathbf{v}) = F(F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)) + \bar{F}(F_1^{-1}(v_1),\ldots,F_d^{-1}(v_d)) = C(\mathbf{u}) + \bar{C}(\mathbf{v})$ yields the asserted weak convergence of $\sqrt{n}(\hat{\beta}_n(\mathbf{u},\mathbf{v}) - \beta(\mathbf{u},\mathbf{v}))$ to a centered tight continuous Gaussian process. □

Remarks. i) A version of the classical Glivenko-Cantelli theorem yields that both empirical functions $\hat{F}_n(\mathbf{x})$ and $\hat{\bar{F}}_n(\mathbf{x})$, as defined in equation (12), are uniformly almost-surely convergent in $\mathbf{x}\in\mathbb{R}^d$. As a consequence we obtain that $C_n(\mathbf{u})$ and $\bar{C}_n(\mathbf{u})$, respectively, are strongly consistent estimators for $C(\mathbf{u})$ and $\bar{C}(\mathbf{u})$ for every $\mathbf{u}\in[0,1]^d$. Thus, $\hat{\beta}_n(\mathbf{u},\mathbf{v})$ is also a strongly consistent estimator for $\beta(\mathbf{u},\mathbf{v})$ for every $(\mathbf{u},\mathbf{v})\in D$.
ii) Weak convergence of the empirical (tail) copula process on the space D_ε with $\varepsilon = 0$ is addressed in Schmidt and Stadtmüller (2006).

The special case $\beta = \beta(\mathbf{1/2},\mathbf{1/2})$ is considered in the next corollary, where we state the asymptotic distribution of its estimator.

Corollary 3 *Let $\mathbf{X}$ have a continuous d-dimensional distribution function F with copula C. Consider the following estimator for $\beta = \beta(\mathbf{1/2},\mathbf{1/2})$*

$$\hat{\beta}_n = \hat{\beta}_n(\mathbf{1/2},\mathbf{1/2}) = h_d\left\{C_n\left(\mathbf{1/2}\right) + \bar{C}_n\left(\mathbf{1/2}\right) - 2^{1-d}\right\}. \tag{16}$$

Under the assumptions that the i-th partial derivatives D_iC and $D_i\bar{C}$ exist and are continuous at the point $\mathbf{1/2}$, we have

$$\sqrt{n}\left(\hat{\beta}_n - \beta\right) \xrightarrow{d} Z \quad \textit{with} \quad Z \sim N(0, \sigma^2)$$

and variance $\sigma^2 = E\{\mathbb{G}(\mathbf{1/2}, \mathbf{1/2})^2\}$ as defined in Theorem 2. Thus, the asymptotic variance σ^2 involves the following expressions

$$\begin{aligned}
E\left\{\mathbb{B}_C\left(\mathbf{1/2}\right)\mathbb{B}_C\left(\mathbf{1/2}\right)\right\} &= C\left(\mathbf{1/2}\right) - \left\{C\left(\mathbf{1/2}\right)\right\}^2 \\
E\left\{\mathbb{B}_C\left(\mathbf{1/2}\right)\mathbb{B}_C\left(\mathbf{1/2}\right)^{(i)}\right\} &= \frac{1}{2}C\left(\mathbf{1/2}\right) \\
E\left\{\mathbb{B}_C\left(\mathbf{1/2}\right)^{(i)}\mathbb{B}_C\left(\mathbf{1/2}\right)^{(j)}\right\} &= \begin{cases} \frac{1}{2} - \left(\frac{1}{2}\right)^2 & i = j \\ C_{ij}\left(\frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 & i \neq j \end{cases}
\end{aligned}$$

$$E\left\{\mathbb{B}_C\left(\mathbf{1/2}\right)\mathbb{B}_{\bar{C}}\left(\mathbf{1/2}\right)\right\} = -C\left(\mathbf{1/2}\right)\bar{C}\left(\mathbf{1/2}\right) \qquad E\left\{\mathbb{B}_C\left(\mathbf{1/2}\right)\mathbb{B}_{\bar{C}}\left(\mathbf{1/2}\right)^{(j)}\right\} = -\frac{1}{2}C\left(\mathbf{1/2}\right).$$

Here, C_{ij} denotes the marginal copula which refers to the variables X_i and X_j for $i \neq j$ and $i, j \in \{1, ..., d\}$.

One advantage of the considered multivariate versions of Blomqvist's β is that the asymptotic variance of their estimators can explicitly be computed whenever the copula and its partial derivatives take an explicit form. For other rank-based dependence measures, such as Kendall's τ or Spearman's ρ, this is rarely possible even in the two-dimensional setting, see Schmid and Schmidt (2006a) for some special cases.

The estimation of the asymptotic variance in Theorem 2 may not be straightforward if the copula is given in implicit form or is completely unknown. This is because the variance involves the copula and its partial derivatives. One may either estimate the copula and its partial derivatives via the empirical copula or apply a nonparametric bootstrap procedure. The following theorem shows that the bootstrap works. In this context, let $(\mathbf{X}_j^B)_{j=1,\ldots,n}$ denote the bootstrap sample which is obtained by sampling from $(\mathbf{X}_j)_{j=1,\ldots,n}$ with replacement. In Section 5.4 we illustrate the performance of the bootstrap using a simulation study.

Theorem 4 (The bootstrap) *Let $\hat{\beta}_n(\mathbf{u}, \mathbf{v})$ be the estimator defined in formula (9) and denote by $\hat{\beta}_n^B(\mathbf{u}, \mathbf{v})$ the corresponding estimator for the bootstrap sample $(\mathbf{X}_j^B)_{j=1,\ldots,n}$. Then, under the assumptions of Theorem 2, $\sqrt{n}\{\hat{\beta}_n^B(\mathbf{u}, \mathbf{v}) - \hat{\beta}_n(\mathbf{u}, \mathbf{v})\}$ converges weakly to the same limit as $\sqrt{n}\{\hat{\beta}_n(\mathbf{u}, \mathbf{v}) - \beta(\mathbf{u}, \mathbf{v})\}$ with probability one. Weak convergence takes place in $\ell^\infty(D_\varepsilon)$.*

Proof. According to Theorem 3.6.2 in Van der Vaart and Wellner (1996), p.347, the empirical process $H_n = \sqrt{n}(\hat{F}_n - F)$ and its bootstrapped version $H_n^B = \sqrt{n}(\hat{F}_n^B - \hat{F}_n)$ converge to the same limit with probability one. The weak convergence of H_n^B and H_n takes place in $\ell^\infty([-\infty, \infty]^d)$. By mimicking the proof of Theorem 2 one obtains the asserted weak convergence of $\sqrt{n}\{\hat{\beta}_n^B(\mathbf{u}, \mathbf{v}) - \hat{\beta}_n(\mathbf{u}, \mathbf{v})\}$. □

4.3 The radial symmetric case $C(\mathbf{u}) = \bar{C}(\mathbf{1} - \mathbf{u})$

In formula (3) it was shown that $\beta = \beta(\mathbf{1/2}, \mathbf{1/2})$ takes the form $\beta = 4C(\mathbf{1/2}) - 1$ in the two-dimensional case $d = 2$. For higher dimensions $d \geq 3$, this relationship is generally not true. However, if the copula is radially symmetric, i.e. $C(\mathbf{u}) = \bar{C}(\mathbf{1} - \mathbf{u})$, the population version of $\beta(\mathbf{u}, \mathbf{v})$ is of the form

$$\beta(\mathbf{u}, \mathbf{v}) = h_d(\mathbf{u}, \mathbf{v})\left\{C\left(\mathbf{u}\right) + C\left(\mathbf{1} - \mathbf{v}\right) - g_d(\mathbf{u}, \mathbf{v})\right\}. \tag{17}$$

Note that especially in higher dimensions it can be tedious to calculate the survival function $\bar{C}$. Also radial symmetry implies that the asymptotic variance of the corresponding estimator can be expressed using the copula C only.

Examples of radially symmetric copulas are the families of elliptical copulas such as the normal or the Student's t-copula, the bivariate Archimedean Frank copula or the independence copula, see Section 5. The following multidimensional extension of Theorem 2.7.3 in Nelsen (2006) provides a method to check for radial symmetry of a copula.

Proposition 5 *Let $\mathbf{X}$ be a d-dimensional random vector with joint distribution function F and copula C. Suppose the marginal random variables X_i are continuous and symmetric about a_i, $i = 1, \ldots, d$. Then $\mathbf{X}$ is radially symmetric about $\mathbf{a} = (a_1, \ldots, a_d)$, i.e. $F(\mathbf{a} + \mathbf{x}) = \bar{F}(\mathbf{a} - \mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^d$, if and only if C is radially symmetric.*

5 Examples

A number of copulas are considered now for which we calculate Blomqvist's multivariate β and the asymptotic variance of its nonparametric estimator.

5.1 Independence

Consider a random vector $\mathbf{X} = (X_1, ..., X_d)$ where the $X_1, ..., X_d$ are stochastically independent (but not necessarily identically distributed). The related copula is the independence copula Π.

Proposition 6 *Let C be the independence copula Π. Then, the asymptotic variance $\sigma^2_{\mathbf{u},\mathbf{v}} := E\{\mathbb{G}(\mathbf{u},\mathbf{v})^2\}$ as given in Theorem 2, for $\mathbf{u} = \mathbf{1/2}$ and $\mathbf{v} = \mathbf{1/2}$ is of the form*

$$\sigma^2_{\mathbf{1/2},\mathbf{1/2}} = \frac{1}{2^{d-1} - 1} = \sigma^2_\star\,, \tag{18}$$

where $\sigma^2_\star$ denotes the asymptotic variance stated in Proposition 1. Thus, the asymptotic variances of the respective estimators under known margins and under unknown margins coincide.
The asymptotic variance for general $(\mathbf{u},\mathbf{v}) \in D_\varepsilon$, $\varepsilon > \mathbf{0}$, is

$$\sigma^2_{\mathbf{u},\mathbf{v}} = h_d(\mathbf{u},\mathbf{v})\Big(\bar{u} + \bar{v} - (\bar{u}+\bar{v})^2 - 2\sum_{i=1}^{d}\Big[\bar{u}_i\{\bar{u}(1-u_i) - \bar{v}u_i\} - \bar{v}_i\{\bar{u}(1-v_i) - \bar{v}v_i\}\Big]\Big) \tag{19}$$

with $\bar{u} := \prod_{i=1}^d u_i$, $\bar{v} := \prod_{i=1}^d (1-v_i)$, $\bar{u}_j := \prod_{i\neq j} u_i$, and $\bar{v}_j := \prod_{i\neq j}(1-v_i)$.

Proof. The asymptotic variances are directly calculated utilizing the facts that $D_iC(u_1, ..., u_d) = \prod_{j\neq i} u_j$ and that the independence copula is radially symmetric, i.e. $\bar{C}(\mathbf{u}) = C(\mathbf{1} - \mathbf{u})$. □

5.2 Elliptical copulas

Elliptical copulas are the copulas of elliptically contoured distributions such as the multivariate normal distributions, Student's t-distributions, symmetric generalized hyperbolic distributions, or α-stable distributions. A d-dimensional random vector $\mathbf{X}$ possesses an elliptically contoured distribution with parameters $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d\times d}$ if

$$\mathbf{X} \stackrel{d}{=} \mu + A'\mathbf{Y}, \tag{20}$$

where $A \in \mathbb{R}^{m\times d}$ with $A'A = \Sigma$, rank(Σ)=m, and $\mathbf{Y}$ is an m-dimensional ($m \in \mathbb{N}$) spherically distributed random vector, i.e., $O\mathbf{Y} \stackrel{d}{=} \mathbf{Y}$ for every orthogonal matrix $O \in \mathbb{R}^{m\times m}$. It can be shown that $\mathbf{Y}$ belongs to the class of spherical distributions if and only if its characteristic function $\phi(t)$, $t \in \mathbb{R}^d$, takes the form $\phi(t) = \Upsilon(t't)$ for some function $\Upsilon : \mathbb{R} \to \mathbb{R}$. Thus, the family of d-dimensional elliptically distributed random vectors is characterized by a location vector μ, a dispersion matrix Σ, and a characteristic generator Υ. We say $\mathbf{X}$ and its copula, respectively, are non-degenerated if the dispersion matrix Σ is positive definite. For a review of elliptically contoured distribution see Fang et al. (1990).

The next proposition justifies the usage of the results in Section 4.3 for the family of elliptical copulas.

Proposition 7 *Every non-degenerated d-dimensional elliptical copula is radially symmetric.*

Proof. Let C be a non-degenerated d-dimensional elliptical copula C. An easy extension of Theorem 2.4.3 in Nelsen (2006) implies that every copula is invariant with respect to strictly increasing transformations of the one-dimensional margins. Thus, we may associate with C a d-dimensional standardized elliptically distributed random vector $\mathbf{X}$, i.e. $\mu = \mathbf{0}$ and $\Sigma_{ii} = 1$ for all $i = 1, \ldots, d$. In particular, each one-dimensional marginal random variable X_i is symmetric about 0. Combining Proposition 5 and the fact that $\mathbf{X}$ has the same distribution as $-\mathbf{X}$, we conclude that C is radially symmetric. $\square$

The following proposition comprises formulas for Blomqvist's multivariate $\beta = \beta(\mathbf{1/2}, \mathbf{1/2})$ for the family of elliptical copulas.

Proposition 8 *Let $\mathbb{F}$ be the family of d-dimensional distributions having a non-degenerated elliptical copula which is characterized by the dispersion matrix $\Sigma = (\Sigma_{ij})$ and the characteristic generator Υ. Then, Blomqvist's $\beta = \beta(\mathbf{1/2}, \mathbf{1/2})$ is invariant with respect to the characteristic generator Υ within this family $\mathbb{F}$. Further, (bivariate) Blomqvist's β between the i-th and j-th marginal distribution of some $F \in \mathbb{F}$ - denoted by β_{ij} - takes the form*

$$\beta_{ij} = 2\arcsin(\varrho_{ij})/\pi, \tag{21}$$

where $\varrho_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$. For every trivariate $(d = 3)$ distribution function $F \in \mathbb{F}$ we have

$$\beta = \frac{2}{3\pi}\big\{\arcsin(\varrho_{12}) + \arcsin(\varrho_{13}) + \arcsin(\varrho_{23})\big\}.$$

In the equi-correlated case $\varrho = \varrho_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}},\ i \neq j$, we have

$$\text{for } d = 4: \quad \beta = \frac{12}{7\pi}\arcsin(\varrho) + \frac{24}{7\pi^2}\int_0^{\varrho} \arcsin\Big(\frac{t}{1+2t}\Big)\frac{dt}{\sqrt{1-t^2}},$$

$$\text{for } d = 5: \quad \beta = \frac{4}{3\pi}\arcsin(\varrho) + \frac{24}{3\pi^2}\int_0^{\varrho} \arcsin\Big(\frac{t}{1+2t}\Big)\frac{dt}{\sqrt{1-t^2}}.$$

For a more general correlation structure $\varrho_{ij} = \delta_i\delta_j,\ |\delta_i| \leq 1$ and general dimension d, we have

$$\beta = \Big\{2^d \int_{-\infty}^{\infty} \prod_{i=1}^{d} \Phi\Big(-\frac{\delta_i t}{\sqrt{1-\delta_i^2}}\Big)\ \varphi(t)\ dt - 1\Big\}\Big/(2^{d-1} - 1)$$

with standard normal distribution function Φ and corresponding density φ.

Proof. Let C be a non-degenerated elliptical copula which is associated to some random vector $\mathbf{X}$. According to the multivariate version of Theorem 2.4.3 in Nelsen (2006), we may assume that $\mathbf{X}$ has a centered elliptically contoured distribution with stochastic representation as given in formula (20). The corresponding spherical random vector $\mathbf{Y}$ has the representation $\mathbf{Y} \stackrel{d}{=} R\mathbf{U}$ - see Theorem 2.2 in Fang et al. (1990) - where the random variable $R \geq 0$ is independent of the d-dimensional random vector $\mathbf{U}$ which is uniformly distributed on the unit sphere $\mathbb{S}_2^{d-1}$ in $\mathbb{R}^d$. Then, $P(\mathbf{X} \leq \mathbf{0}) = \int_0^\infty P(rA'\mathbf{U} \leq 0)\ dF_R(r) = P(A'\mathbf{U} \leq 0)$ which is invariant with respect to the random variable R. This invariance holds also with respect to the characteristic generator Υ. Combining this fact with Proposition 7 and $C(\mathbf{1/2}) = P(\mathbf{X} \leq \mathbf{0})$, yields the first assertion of the proposition. The explicit formulas of β are now calculated by using the orthant probabilities of multivariate normal distributions, as given in Steck (1962) and Kotz et al. (2000). $\square$

Remark. Theorem 3.1 in Fang et al. (2002) and formula (21) imply that the population version of Kendall's τ coincides with Blomqvist's β within the family of bivariate distributions having an elliptical copula - so called meta-elliptical distributions.

Next we derive an explicit form for the asymptotic variance of Blomqvist's beta for the family of elliptical copulas and provide a variance stabilizing transformation.

Proposition 9 *Let* $\mathbf{X}$ *be a* d*-dimensional random vector with non-degenerated elliptical copula which fulfills the prerequisites in Corollary 3. Denote by* $\hat{\beta}_n^{(ij)} = \hat{\beta}_n^{(ij)}(\mathbf{1/2}, \mathbf{1/2})$ *the estimator of Blomqvist's* β_{ij} *between the* i*-th and* j*-th margin of* $\mathbf{X}$*. The corresponding asymptotic variance* $\sigma^2(\varrho_{ij})$ *can be expressed by*

$$\sigma^2(\varrho_{ij}) = \frac{\pi^2 - 4\{\arcsin(\varrho_{ij})\}^2}{\pi^2} = 1 - \beta_{ij}^2$$

with $\varrho_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$*. This yields a variance stabilizing transformation* $g(t) = \arcsin(t)$ *which implies*

$$\sqrt{n}\left\{g(\hat{\beta}_n^{(ij)}) - g(\beta_{ij})\right\} \xrightarrow{d} N(0,1).$$

For the *proof* calculate the asymptotic variance, as given in Corollary 3, and utilize the radial symmetry stated in Proposition 7. Note that for any bivariate elliptical copula C we have $D_iC(\mathbf{u})|_{\mathbf{u}=\mathbf{1/2}} = 1/2$. The variance stabilizing transformation is useful for the construction of confidence bands for Blomqvist's β. In particular, the asymptotic confidence interval for Blomqvist's β_{ij} in Proposition 9 is given by

$$(\sin\{\arcsin(\beta_{ij}) - z_{1-\alpha}/\sqrt{n}\}, \sin\{\arcsin(\beta_{ij}) + z_{1-\alpha}/\sqrt{n}\}),$$

where $z_{1-\alpha}$ is such that $\Phi(z_{1-\alpha}) = 1 - \alpha$ with standard normal distribution Φ. Note that beside stabilization of the variance, the above transformation also symmetrizes the distribution of the sample version of Blomqvist's β. This may be compared with Fisher's z-transformation of Pearson's sample correlation-coefficient which has the form $g(t) = \operatorname{arctanh}(t)$.

5.3 Archimedean copulas

A d-dimensional *Archimedean* copula is of the form

$$C(\mathbf{u}) = C(u_1, \ldots, u_d) = \varphi^{-1}\{\varphi(u_1) + \cdots + \varphi(u_d)\}$$

with continuous, strictly decreasing *generator* $\varphi : [0,1] \to [0,\infty]$ such that $\varphi(1) = 0$, $\varphi(0) = \infty$, and the inverse function φ^{-1} is completely monotonic on $[0,\infty)$. For more details we refer to Joe (1997), Chapter 4.

Table 1 provides the asymptotic variance of the estimator $\hat{\beta}_n = \hat{\beta}_n(\mathbf{1/2}, \mathbf{1/2})$, as given in Corollary 3, for a number of bivariate Archimedean copulas. Similar results are obtained for the general estimator $\hat{\beta}_n(\mathbf{u}, \mathbf{v})$ but are omitted for presentational reasons. Figure 2 illustrates the range of the asympotic variance $\sigma^2(\theta)$ depending on the copula parameter θ. For the bivariate Frank copula - No. (5) in Table 1 - we may utilize the findings of Section 4.3 since this copula is radially symmetric. However, the next theorem shows that, for $d \geq 3$, there exists no radially symmetric Archimedean copula.

Theorem 10 *There exists no radially symmetric* d*-dimensional Archimedean copula for* $d \geq 3$.

Table 1: Asymptotic variance $\sigma^2(\theta)$, as given in Corollary 3, for selected Archimedean copulas. No. (1) is the Clayton copula, (3) corresponds to the Ali-Mikhail-Haq copula, (4) is the Gumbel-Hougaard copula, (5) denotes the Frank copula, and (6) is the Gumbel-Barnett copula. We restrict the domain of the parameter θ such that the copula is well defined and the asymptotic variance $\sigma^2(\theta) > 0$.

No.	$C(u_1, u_2)$	Parameters	$\sigma^2(\theta)$
(1)	$\Big\{ \max(u_1^{-\theta} + u_2^{-\theta} - 1, 0) \Big\}^{-1/\theta}$	$\theta \in (-1, \infty)\backslash\{0\}$	$8h_\theta\{1 - 2h_\theta + (4h_\theta^{\theta+1}2^\theta - 1)^2\}$ $h_\theta = (2^{\theta+1} - 1)^{-1/\theta}$
(2)	$\max\Big[1 - \Big\{(1-u_1)^\theta + (1-u_2)^\theta\Big\}^{1/\theta}, 0\Big]$	$\theta \in [1, \infty)$	$24h_\theta - 16h_\theta^2 - 20g_\theta + 16g_\theta^2 - 4g_\theta^3$ $h_\theta = 2^{(1-\theta)/\theta}, \quad g_\theta = 2^{1/\theta}$
(3)	$\dfrac{u_1 u_2}{1 - \theta(1-u_1)(1-u_2)}$	$\theta \in [-1, 1)$	$16\dfrac{\theta^4 - 7\theta^3 + 36\theta^2 - 80\theta + 64}{(4-\theta)^5}$
(4)	$\exp\Big[- \{(-\ln u_1)^\theta + (-\ln u_2)^\theta\}^{1/\theta}\Big]$	$\theta \in [1, \infty)$	$8h_\theta\{1 - 2h_\theta + (2^{1/\theta+1}h_\theta - 1)^2\}$ $h_\theta = \exp(-2^{1/\theta}\ln 2)$
(5)	$-\dfrac{1}{\theta}\ln\Big\{1 + \dfrac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\Big\}$	$\theta \in \mathbb{R}\backslash\{0\}$	$\dfrac{16}{\theta^2}\ln\Big(\dfrac{2}{1+e^{\theta/2}}\Big)\ln\Big(\dfrac{1+e^{-\theta/2}}{2}\Big)$
(6)	$u_1 u_2 \exp(-\theta \ln u_1 \ln u_2)$	$\theta \in (0, 1]$	$h_\theta\{2 - h_\theta + 2(h_\theta + \theta h_\theta \ln 2 - 1)^2\}$ $h_\theta = \exp\{-\theta(\ln 2)^2\}$
(7)	$\Big[1 + \Big\{(u_1^{-1} - 1)^\theta + (u_2^{-1} - 1)^\theta\Big\}^{1/\theta}\Big]^{-1}$	$\theta \in [1, \infty)$	$\dfrac{8}{h_\theta}\Big\{1 - \dfrac{2}{h_\theta} + \Big(\dfrac{2^{2+1/\theta}}{h_\theta^2} - 1\Big)\Big\}$ $h_\theta = 1 + 2^{1/\theta}$
(8)	$\theta / \ln\big(e^{\theta/u_1} + e^{\theta/u_2} - e^\theta\big)$	$\theta \in (0, \infty)$	$\dfrac{8\theta}{\ln h_\theta}\Big\{1 - \dfrac{16\theta}{\ln h_\theta} + \Big(\dfrac{8\theta^2 e^{2\theta}}{(\ln h_\theta)^2 h_\theta} - 1\Big)^2\Big\}$ $h_\theta = 2\exp(2\theta) - \exp(\theta)$

Proof. Suppose C is a d-dimensional radially symmetric copula ($d \geq 3$) which is of Archimedean type. Then, the bivariate marginal copula $C_2(u_1, u_2) = C(u_1, u_2, 1, \dots, 1)$ must also be radially symmetric due to the fact that C is radially symmetric if and only if

$$C(\mathbf{1} - \mathbf{u}) = \bar{C}(\mathbf{u}) = \sum_{k=0}^{d} (-1)^k \sum_{\substack{I \subset \{1,\dots,d\} \\ |I| = k}} C\left(\mathbf{u}^I\right),$$

where the components of $\mathbf{u}^I$ are such that $u_i^I = u_i$ if $i \in I$ and $u_i^I = 1$ if $i \notin I$. Further, if C is Archimedean than C_2 is also an Archimedean copula with the same generator function φ. According to Frank (1979), the only bivariate Archimedean copula which is radially symmetric is the Frank copula. Consequently, the only d-dimensional Archimedean copula which might be radially symmetric must be the Frank copula (see the example below for the explicit form). For $d = 3$, this copula yields

$$C(\mathbf{1/2}) - \bar{C}(\mathbf{1/2}) = \frac{1}{2} + \frac{3}{\theta}\ln\left\{1 + \frac{(e^{-\theta/2} - 1)^2}{(e^{-\theta} - 1)}\right\} - \frac{2}{\theta}\ln\left\{1 + \frac{(e^{-\theta/2} - 1)^3}{(e^{-\theta} - 1)^2}\right\}, \qquad \theta > 0.$$

This expression is zero if and only if θ is zero (in limit). Since $\theta > 0$, we conclude that $C(\mathbf{1/2}) \neq \bar{C}(\mathbf{1/2})$ and, thus, the three-dimensional Frank copula is not radially symmetric. This implies that

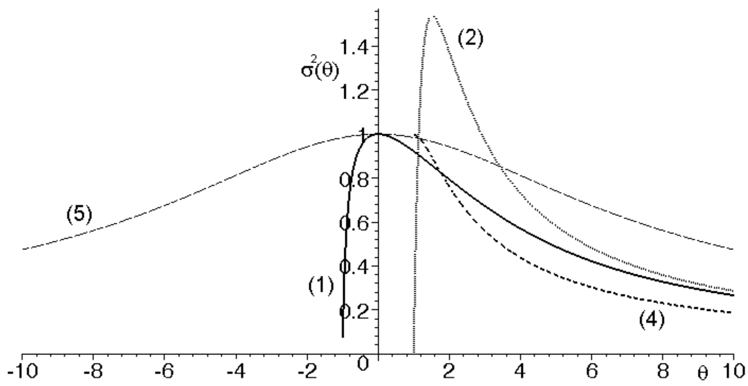

Figure 2: Asymptotic variance $\sigma^2(\theta)$ of Blomqvist's $\hat{\beta}_n$ as given in Corollary 3, for various copulas numbered as in Table 1.

every d-dimensional Frank copula for $d \geq 3$ is not radially symmetric, because otherwise every lower dimensional copula would also be radially symmetric. $\square$

Examples. The family of d-dimensional *Frank copulas* ($d \geq 2$) is given by

$$C_\theta(\mathbf{u}) = -\frac{1}{\theta} \ln \left\{ 1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1) \cdots (e^{-\theta u_d} - 1)}{(e^{-\theta} - 1)^{d-1}} \right\} \quad \text{with } \theta > 0$$

and the family of d-dimensional *Gumbel-Hougaard copulas* ($d \geq 2$) is given by

$$C_\theta(\mathbf{u}) = \exp\left[- \left\{ (-\ln u_1)^\theta + (-\ln u_2)^\theta + \cdots + (-\ln u_d)^\theta \right\}^{1/\theta} \right], \quad \text{with } \theta \geq 1.$$

Figure 3 shows the asymptotic standard deviation of $\sqrt{n}\{\hat{\beta}_n(\mathbf{1/2}, \mathbf{1}) - \beta(\mathbf{1/2}, \mathbf{1})\}$ for various dimensions d for the above copulas. The independence copula Π corresponds to the parameter $\theta = 0$ (Frank copula) and $\theta = 1$ (Gumbel-Hougaard copula). Figure 3 illustrates that, in the independence case, the asymptotic standard deviation of the estimator decreases with increasing dimension d. It can be shown that this property does not hold for any fixed $\theta > 0$ (Frank copula) and $\theta > 1$ (Gumbel-Hougaard copula).

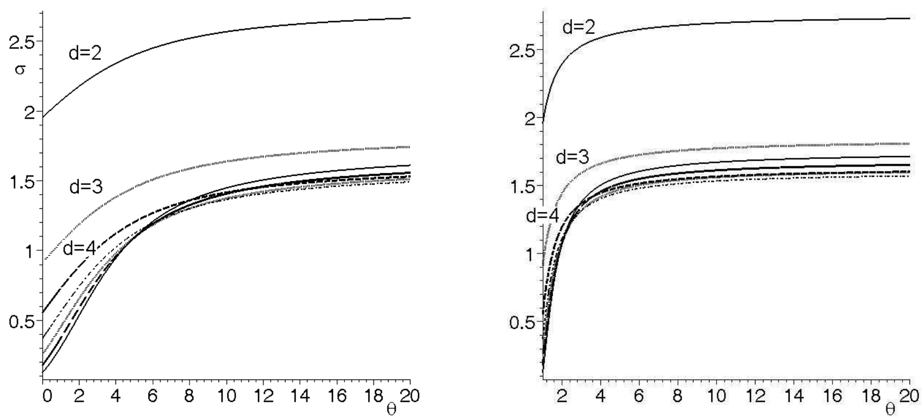

Figure 3: Asymptotic standard deviation of $\sqrt{n}\{\hat{\beta}_n(\mathbf{1/2}, \mathbf{1}) - \beta(\mathbf{1/2}, \mathbf{1})\}$ for the d-dimensional Frank copula (left plot) and Gumbel-Hougaard copula (right plot) for dimensions $d = 2, \ldots, 8$.

5.4 Simulation Study

In the following we examine the finite sample behavior of the sample version of Blomqvist's multivariate β. In particular, we are interested in the performance of the nonparametric bootstrap for the estimation of the asymptotic variance. We confine ourselves to the estimation of $\beta = \beta(\mathbf{1/2}, \mathbf{1/2})$ due to limited space.

Two copulas are considered in various dimensions d, namely, the Gaussian copula and the Student's t-copula. These copulas are the most popular copulas for applications in financial engineering, see Cherubini et al. (2004) for an overview. The d-dimensional equi-correlated Gaussian copula - considered in Table 2 - is given by

$$C(u_1, ..., u_d; \varrho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \dots \int_{-\infty}^{\Phi^{-1}(u_d)} (2\pi)^{-\frac{d}{2}} det\{\Sigma(\varrho)\}^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}\mathbf{x}'\Sigma(\varrho)^{-1}\mathbf{x}\Big)\, dx_d \dots dx_1$$

where $\Sigma(\varrho) = \varrho\mathbf{1}\mathbf{1}' + (1-\varrho)I_d$ with identity matrix I_d and $-\frac{1}{d-1} < \varrho < 1$. The Student's t-copula represents the copula of the multivariate Student's t-distribution and is defined similarly.

Tables 2 and 3 summarize the simulation results for the above copulas for dimensions $d = 2$, 5, and 10. Two three-dimensional Student's t-copulas, which are not equi-correlated, are considered in Table 4. The first and second column in every table contain values of the parameters and sample sizes, respectively. The third column contains the true value of Blomqvist's β in the two- and three-dimensional cases, whereas it contains an approximation of Blomqvist's β for dimensions $d = 5, 10$, cf. Proposition 8. The latter values of Blomqvist's multivariate β have been derived by computing $\hat{\beta}$ - the index n will be suppressed for notational convenience - from samples of length 500,000. The first two digits behind the decimal point are accurate. The fourth and sixth columns contain the means $m(\hat{\beta})$ and the standard deviations $\hat{\sigma}(\hat{\beta})$ of $\hat{\beta}$ over 300 Monte Carlo replications.

Comparing the third and fourth column in every table (β and $m(\hat{\beta})$), we observe a minor bias for small sample sizes, such as $n = 100$, in every dimension under study. There is good agreement between the fourth and fifth column, i.e., between $m(\hat{\beta})$ and $m(\hat{\beta}^B)$. The sixth column shows the standard error $\hat{\sigma}(\hat{\beta})$ of $\hat{\beta}$ which is decreasing with increasing sample size n. The amount of $\hat{\sigma}(\hat{\beta})$ depends on the copula, its parameters, and the dimension d. The seventh column contains the empirical mean $m(\hat{\sigma}^B)$ of the bootstrap estimates. The good agreement between the sixth and seventh column indicates that the nonparametric bootstrap works well for every dimension under study, although a bias is present for small sample sizes. Column 8 shows that the standard deviation of $\hat{\sigma}^B$ over 300 Monte Carlo replications is low, especially for $n = 500$ and 1000. For comparison, the last column provides the estimates of the asymptotic standard deviation given in Corollary 3. It can be seen that the asymptotic standard deviation is well estimated by $\hat{\sigma}^B\sqrt{n}$ for both copulas and every parameter constellation under study, even for the small sample size $n = 100$.

6 Tail dependence

The concept of tail dependence helps to analyze and to model dependencies between extreme events. For example in finance (see Poon et al. (2004) for an overview) tail dependent distributions or copulas are frequently used in order to model the possible dependencies between large negative asset-returns or portfolio losses. The *lower tail-dependence coefficient* λ_L between two random variables X_1 and X_2 with copula C is defined by

$$\lambda_L := \lim_{p\downarrow 0} P(X_1 \le F_1^{-1}(p) \mid X_2 \le F_2^{-1}(p)) = \lim_{p\downarrow 0} \frac{C(p,p)}{p}, \tag{22}$$

if the limit exists. If $\lambda_L > 0$, we say that $\mathbf{X} = (X_1, X_2)'$ is tail dependent, otherwise $\mathbf{X}$ is tail independent. This dependence measure was introduced by Sibuya (1960) and plays a role in bivariate extreme value theory. For the independence copula $\Pi(u, v)$ we have $\lambda_L = 0$ (tail independence) and for the comonotonic copula $M(u, v)$ we have $\lambda_L = 1$ (tail dependence). Note that the tail-dependence coefficient λ_L is a copula-based dependence measure. Furthermore, it is related to the

Table 2: **Gaussian copula.** Simulation results for bootstrapping Blomqvist's multivariate $\hat{\beta}_n$ (the index n is suppressed). Results are based on 300 samples with sample size n generated from a d-variate Gaussian copula with equi-correlation parameter ϱ. The columns provide the empirical means - denoted by $m()$ - and the empirical standard deviations - denoted by $\hat{\sigma}()$ - based on the simulated data and the respective bootstrap samples. The statistics with superscript B refer to the bootstrap sample. 250 bootstrap replications were drawn from each sample. The empirical standard deviation of the bootstrapped statistics is abbreviated by $\hat{\sigma}^B = \hat{\sigma}(\hat{\beta}^B)$.

ϱ	n	β	$m(\hat{\beta})$	$m(\hat{\beta}^B)$	$\hat{\sigma}(\hat{\beta})$	$m(\hat{\sigma}^B)$	$\hat{\sigma}(\hat{\sigma}^B)$	$\hat{\sigma}(\hat{\beta})\sqrt{n}$
Dimension $d = 2$								
0	100	0	.002	.005	.102	.110	.006	1.019
	500	0	.000	.000	.046	.047	.002	1.021
	1000	0	-.001	-.002	.032	.033	.001	1.016
0.3	100	.194	.194	.191	.092	.109	.006	.920
	500	.194	.193	.193	.047	.046	.002	1.041
	1000	.194	.194	.194	.032	.032	.002	1.002
0.7	100	.494	.486	.485	.083	.099	.007	.828
	500	.494	.494	.494	.037	.041	.003	.834
	1000	.494	.494	.493	.027	.029	.001	.841
Dimension $d = 5$								
0	100	0	.001	.001	.026	.029	.005	.257
	500	0	.000	.000	.012	.012	.001	.264
	1000	0	.000	.000	.008	.009	.001	.249
0.3	100	.156	.155	.154	.042	.048	.004	.421
	500	.156	.156	.156	.020	.020	.001	.441
	1000	.156	.156	.156	.012	.014	.001	.393
0.7	100	.452	.449	.445	.052	.059	.004	.524
	500	.452	.452	.452	.025	.025	.001	.555
	1000	.452	.452	.451	.017	.017	.001	.540
Dimension $d = 10$								
0	100	0	.000	.000	.005	.004	.004	.046
	500	0	.000	.000	.002	.002	.001	.046
	1000	0	.000	.000	.001	.001	.001	.043
0.3	100	.071	.069	.068	.025	.028	.004	.254
	500	.071	.071	.071	.012	.012	.001	.264
	1000	.071	.071	.071	.007	.009	.001	.236
0.7	100	.346	.341	.338	.047	.052	.003	.470
	500	.346	.344	.344	.022	.022	.001	.481
	1000	.346	.346	.345	.015	.015	.001	.469

definition of the multivariate version of Blomqvist's $\beta(\mathbf{u}, \mathbf{v})$ given in formula (3). More precisely, for $d = 2$ we have

$$\lambda_L = \lim_{p \downarrow 0} \beta\{(p, p), \mathbf{1}\}(1 - p) + p = \lim_{p \downarrow 0} \beta\{(p, p), \mathbf{1}\} \tag{23}$$

if the limit exists. The latter limit exists if and only if the limit in equation (22) exists. Formula (23) motivates the following definition of a lower multivariate tail-dependence coefficient:

$$\lambda_L := \lim_{p \downarrow 0} \beta\{(p, \ldots, p), \mathbf{1}\},$$

if the limit exists. This measure of tail dependence describes the amount of dependence in the lower tail of the copula function, e.g. it may measure the degree of comovement between extremely negative stock returns.

Let us consider the family of elliptical copulas. Note that the density function f of an elliptically contoured distribution, if it exists, can be represented by

$$f(\mathbf{x}) = |\Sigma|^{-1/2} h\{(\mathbf{x} - \mu)' \Sigma^{-1} (\mathbf{x} - \mu)\}, \qquad \mathbf{x} \in \mathbb{R}^d, \tag{24}$$

Table 3: **Student's t-copula.** Simulation results for bootstrapping Blomqvist's multivariate $\hat{\beta}_n$ (the index n is suppressed). Results are based on 300 samples with sample size n generated from a d-variate Student's t-copula with 1 degree of freedom and equi-correlation parameter ϱ. The columns provide the empirical means - denoted by $m()$ - and the empirical standard deviations - denoted by $\hat{\sigma}()$ - based on the simulated data and the respective bootstrap samples. The statistics with superscript B refer to the bootstrap sample. 250 bootstrap replications were drawn from each sample. The empirical standard deviation of the bootstrapped statistics is abbreviated by $\hat{\sigma}^B = \hat{\sigma}(\hat{\beta}^B)$.

ϱ	n	β	$m(\hat{\beta})$	$m(\hat{\beta}^B)$	$\hat{\sigma}(\hat{\beta})$	$m(\hat{\sigma}^B)$	$\hat{\sigma}(\hat{\sigma}^B)$	$\hat{\sigma}(\hat{\beta})\sqrt{n}$
Dimension $d=2$								
0	100	0	-.005	-.006	.100	.111	.006	1.000
	500	0	.000	.000	.043	.047	.002	.960
	1000	0	-.001	-.002	.033	.033	.002	1.031
0.3	100	.194	.189	.190	.102	.109	.007	1.025
	500	.194	.188	.188	.044	.046	.002	.991
	1000	.194	.195	.195	.030	.032	.002	.935
0.7	100	.494	.479	.476	.089	.100	.007	.886
	500	.494	.490	.490	.036	.041	.002	.811
	1000	.494	.494	.493	.026	.029	.002	.829
Dimension $d=5$								
0	100	0	-.001	-.001	.026	.029	.005	.255
	500	0	.001	.000	.011	.012	.001	.254
	1000	0	.000	.000	.008	.008	.001	.249
0.3	100	.156	.156	.155	.045	.047	.004	.448
	500	.156	.156	.156	.020	.020	.001	.443
	1000	.156	.156	.156	.013	.014	.001	.421
0.7	100	.452	.450	.447	.053	.058	.003	.532
	500	.452	.450	.449	.026	.025	.001	.573
	1000	.452	.452	.452	.016	.017	.001	.519
Dimension $d=10$								
0	100	0	.000	.000	.004	.004	.004	.044
	500	0	.000	.000	.002	.002	.001	.043
	1000	0	.000	.000	.002	.001	.001	.050
0.3	100	.071	.067	.067	.025	.028	.004	.249
	500	.071	.070	.069	.011	.012	.001	.238
	1000	.071	.070	.070	.008	.008	.001	.266
0.7	100	.346	.336	.332	.049	.051	.003	.487
	500	.346	.344	.343	.021	.022	.001	.479
	1000	.346	.345	.344	.015	.015	.001	.470

where $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is called the *density generator*. Thus every elliptical copula is determined by the density generator. If the density generator h is regularly varying, Schmidt (2002) proved the following relationship:

$$\lambda_L = \lim_{p \downarrow 0} \beta\{(p,p),\mathbf{1}\} = \int_0^{f(\varrho)} \frac{u^\alpha}{\sqrt{1-u^2}} du \Big/ \int_0^1 \frac{u^\alpha}{\sqrt{1-u^2}} du,$$

where ϱ denotes the correlation coefficient, $f(\varrho) := \left\{1 + \frac{(1-\varrho)^2}{1-\varrho^2}\right\}^{-1/2}$, and α is the tail index of the density generator h. The next theorem gives a representation of the d-dimensional λ_L for elliptical copulas.

Theorem 11 *Let* $\mathbf{X}$ *be a d-dimensional random vector with copula C which is the copula of an elliptically contoured distribution with density generator h. If the density generator is regularly*

Table 4: **Skewed copulas.** Simulation results for bootstrapping Blomqvist's multivariate $\hat{\beta}_n$ (the index n is suppressed). Results are based on 300 samples with sample size n generated from two types of three-dimensional copulas. The first type corresponds to a three-dimensional Gaussian copula and the second type to a three-dimensional Student's t-copula with 1 degree of freedom. We consider two different constellations of the correlation parameters: I) $\varrho_{12} = 0.9, \varrho_{13} = 0, \varrho_{23} = 0$ and II) $\varrho_{12} = 0.9, \varrho_{13} = 0.3, \varrho_{23} = 0$. The columns provide the empirical means - denoted by $m()$ - and the empirical standard deviations - denoted by $\hat{\sigma}()$ - based on the simulated data and the respective bootstrap samples. The statistics with superscript B refer to the bootstrap sample. 250 bootstrap replications were drawn from each sample. The empirical standard deviation of the bootstrapped statistics is abbreviated by $\hat{\sigma}^B = \hat{\sigma}(\hat{\beta}^B)$.

Parameter constellation	n	β	$m(\hat{\beta})$	$m(\hat{\beta}^B)$	$\hat{\sigma}(\hat{\beta})$	$m(\hat{\sigma}^B)$	$\hat{\sigma}(\hat{\sigma}^B)$	$\hat{\sigma}(\hat{\beta})\sqrt{n}$
Multivariate Gaussian-copula with dimension $d = 3$								
I)	100	.238	.239	.237	.060	.073	.004	.601
	500	.238	.235	.235	.028	.031	.002	.615
	1000	.238	.237	.237	.021	.022	.001	.665
II)	100	.302	.294	.293	.065	.074	.004	.650
	500	.302	.303	.302	.029	.031	.002	.644
	1000	.302	.304	.304	.022	.022	.001	.680
Multivariate Student's t-copula with dimension $d = 3$								
I)	100	.238	.229	.228	.065	.073	.005	.650
	500	.238	.235	.235	.030	.031	.001	.660
	1000	.238	.238	.238	.021	.022	.001	.658
II)	100	.302	.295	.293	.067	.074	.004	.667
	500	.302	.301	.301	.028	.031	.002	.633
	1000	.302	.302	.301	.022	.022	.001	.686

varying (at infinity) with index $-(\alpha + d)/2$, $\alpha > 0$, i.e., $\lim_{t\to\infty} h(ts)/h(t) = s^{-(\alpha+d)/2}$, then

$$\lambda_L = \frac{2}{E(B^\alpha)} \int_{\overline{\mathbb{S}}^{d-1}} \min_i \left\{(\sqrt{\Sigma}\mathbf{a})_i\right\}^\alpha S(d\mathbf{a})$$

with B^2 is Beta$(1/2, (d-1)/2)$ distributed. The space $\overline{\mathbb{S}}^{d-1} := \mathbb{S}^{d-1} \backslash \{\mathbf{a} \mid (\sqrt{\boldsymbol{\Sigma}}\mathbf{a})_\mathbf{i} < \mathbf{0}$ for some $\mathbf{i}\}$, where $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : ||x|| = 1\}$ denotes the $(d-1)$-dimensional unit sphere (regarding the Euclidean norm) and $S(\cdot)$ is the uniform measure on it.

Proof. Without loss of generality we set $\mu = \mathbf{0}$ and the diagonal elements of Σ equal to 1, since elliptical copulas are invariant with respect to these parameters. Then the univariate distributions of the corresponding elliptically contoured distribution function F coincide and we will denote them by G. Let $\mathbf{Y}$ be the random vector associated with F. Note that the left endpoint of G equals $-\infty$ because the density generator h is regularly varying at infinity. We have

$$\lambda_L = \lim_{p\downarrow 0} \beta\{(p, \ldots, p), \mathbf{1}\} = \lim_{p\downarrow 0} \frac{C(\mathbf{p}) - p^d}{p + p^d} = \lim_{p\downarrow 0} \frac{P\{Y_i \le G^{-1}(p), i = 1, \ldots, d\}}{p} = \lim_{t\to\infty} \frac{P(\mathbf{Y} > t\mathbf{1})}{\bar{G}(t)}$$

with $\bar{G} = 1 - G$, due to the radial symmetry of $\mathbf{Y}$. Further, G is continuous and possesses a density if $d \ge 2$ according to Fang et al. (1990), pp.36.

It remains to be shown that

$$\frac{\bar{F}(t\mathbf{1})}{\bar{G}(t)} \to \frac{2}{E(B^\alpha)} \int_{\overline{\mathbb{S}}^{d-1}} \min_i \left\{(\sqrt{\Sigma}\mathbf{a})_i\right\}^\alpha S(d\mathbf{a}), \qquad \text{as } t \to \infty.$$

The random variable B^2 is $Beta(1/2,(d-1)/2)$ distributed. We utilize the following stochastic representation of elliptically contoured distributions with location $\mu = \mathbf{0}$:

$$\mathbf{Y} \stackrel{d}{=} R_d \sqrt{\Sigma}\mathbf{U}^{(d)} \tag{25}$$

with $\sqrt{\Sigma}\sqrt{\Sigma}' = \Sigma$. The random variable $R_d \geq 0$ is stochastically independent of the d-dimensional random vector $\mathbf{U}^{(d)}$ which is uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$. Further, G is the distribution function of a random variable $R_1 U$, where $R_1 \geq 0$ is stochastically independent of the Bernoulli random variable U. Proposition 3.1 in Schmidt (2002) and the related proof imply that $P(R_1 > t)/P(R_d > t) \to E(B^\alpha)$ as $t \to \infty$. For $t \geq 0$ we have

$$\frac{\bar{F}(t\mathbf{1})}{\bar{G}(t)} = \frac{P(R_d\sqrt{\Sigma}\mathbf{U}^{(d)} > t\mathbf{1})}{P(R_1 U > t)} = 2 \cdot \frac{P(R_d > t)}{P(R_1 > t)} \int_{\overline{\mathbb{S}}^{d-1}} \frac{P(R_d\sqrt{\Sigma}\mathbf{a} > t\mathbf{1})}{P(R_d > t)}\, S(d\mathbf{a}). \tag{26}$$

Due to the uniform convergence of $P(R_d > tx)/P(R_d > t) \to x^{-\alpha}$ in $x \in [\varepsilon, \infty)$ for any fixed $\varepsilon > 0$ by Proposition 3.1 of the last reference and Theorem 1.5.2 in Bingham, Goldie, and Teugels (1987), we obtain

$$\frac{P(R_d\sqrt{\Sigma}\mathbf{a} > t\mathbf{1})}{P(R_d > t)} = \frac{P\big[R_d > t \cdot \max_i \big\{1/(\sqrt{\Sigma}\mathbf{a})_i\big\}\big]}{P(R_d > t)} \to \min_i \big\{(\sqrt{\Sigma}\mathbf{a})_i\big\}^\alpha,$$

which converges uniformly in $\mathbf{a} \in \overline{\mathbb{S}}^{d-1}$. Combining this with formula (26) finishes the proof. $\square$

Remarks. i) The family of Student's t-copulas with $\nu = \alpha$ degrees of freedom, which are the copulas of multivariate Student's t-distributions, fulfill the prerequisites in Theorem 11.

ii) Theorem 11 can be generalized to elliptical copulas without an existing density function. More precisely, the theorem holds if the (tail-) distribution function of the generating random variable R_d, as given in the stochastic representation (25), is regularly varying with tail index α. Note that symmetric α-stable distributions with $\alpha < 2$ do not possess a density function but their generating random variable is regularly varying.

7 Asymptotic efficiency

We analyze asymptotic efficiencies of Blomqvist's multivariate $\hat{\beta}_n(\mathbf{u}, \mathbf{v})$ with respect to multivariate sample versions of Spearman's ρ. We choose Spearman's ρ since it is the best-known bivariate rank based dependence measure in social and economic statistics. Various multivariate versions of Spearman's ρ are e.g. discussed in Wolff (1980), Joe (1990), Nelsen (1996), and Schmid and Schmidt (2006a). The following two sample versions are for instance considered in the latter reference:

$$\hat{\rho}_n = h(d) \cdot \Big\{\frac{2^d}{n}\sum_{j=1}^n \prod_{i=1}^d (1-\hat{U}_{ij,n}) - 1\Big\} \quad \text{and} \quad \hat{\rho}'_n = \frac{12}{n}\binom{d}{2}^{-1} \sum_{k<l}\sum_{j=1}^n (1-\hat{U}_{kj,n})(1-\hat{U}_{lj,n}) - 3$$

with $\hat{U}_{ij,n}$ defined in formula (8) and $h(d) = (d+1)/\{2^d - (d+1)\}$. Both estimators coincide for $d = 2$.

7.1 Relative efficiency

The (asymptotic) relative efficiency compares the concentration of the limiting distribution of two sequences of estimators $(T_n^{(1)})$ and $(T_n^{(2)})$, $n \in \mathbb{N}$. If the estimators are asymptotically normal with $\sqrt{n}\{T_n^{(i)} - \mu_i(\theta)\} \xrightarrow{d} N(0, \sigma_i^2(\theta))$ as $n \to \infty$, the relative efficiency of $(T_n^{(1)})$ with respect to $(T_n^{(2)})$ takes the form

$$RE(\theta) = \frac{\sigma_2^2(\theta)}{\sigma_1^2(\theta)} = \lim_{\nu\to\infty} \frac{n_{\nu,2}}{n_{\nu,1}}, \tag{27}$$

where $n_{\nu,i},\ i = 1, 2$, denote the number of observations needed such that $\sqrt{\nu}\{T^{(i)}_{n_{\nu,i}} - \mu_i(\theta)\} \xrightarrow{d} N(0, 1)$ as $\nu \to \infty$. Thus, the quotient in (27) indicates the proportion of observations the second sequence of estimators $(T^{(2)}_n)$ needs in order to achieve the same (asymptotic) precision as $(T^{(1)}_n)$.

Examples. The following copula is called *Kotz-Johnson copula*, see Nelsen (2006), p.82:

$$C(u, v) = uv + \theta_2 uv(1 - u)(1 - v)(1 + \theta_1 uv) \quad \text{for all} \quad -1 \le \theta_1, \theta_2 \le 1. \tag{28}$$

It forms a generalization of the well-known family of Farlie-Gumbel-Morgenstern copulas which arise for $\theta_1 = 0$. The relative efficiency of Blomqvist's $\hat{\beta}_n = \hat{\beta}_n(\mathbf{1/2}, \mathbf{1/2})$ with respect to Spearman's $\hat{\rho}_n$ is shown in the left plot of Figure 4. The asymptotic variance of the latter has been calculated in Schmid and Schmidt (2006a). Further, the right plot in Figure 4 illustrates the logarithm of the relative efficiency of Blomqvist's $\hat{\beta}_n$ with respect to Spearman's $\hat{\rho}_n$ and $\hat{\rho}'_n$ for the d-dimensional independence copula. It shows that for dimension $d = 2$ the relative efficiencies equal one, whereas for dimensions $3 \le d \le 8$ Blomqvist's $\hat{\beta}_n$ outperforms both Spearman's $\hat{\rho}_n$ and $\hat{\rho}'_n$. For $d \ge 9$, Blomqvist's $\hat{\beta}_n$ is better than Spearman's $\hat{\rho}'_n$, whereas it is worse than Spearman's $\hat{\rho}_n$.

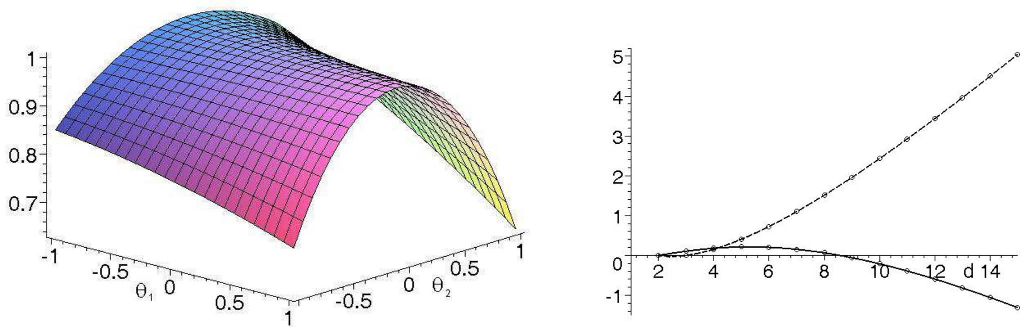

Figure 4: **Left plot.** Relative efficiency of Blomqvist's $\hat{\beta}_n$ with respect to Spearman's $\hat{\rho}_n$ for the family of Kotz-Johnson copulas with parameter $\theta = (\theta_1, \theta_2)$. **Right plot.** Logarithm of the relative efficiency of Blomqvist's $\hat{\beta}_n$ with respect to Spearman's $\hat{\rho}_n$ (solid line) and of Blomqvist's $\hat{\beta}_n$ with respect to the average Spearman's $\hat{\rho}'_n$ (dashed line) for the d-dimensional independence copula Π. The x-axis displays the dimension d.

7.2 Pitman efficiency

The (asymptotic) Pitman efficiency is derived under the assumption that the copula of a random sample takes the form

$$C(\mathbf{u}; \theta) = \prod_{i=1}^{d} u_i + \theta \Psi(\mathbf{u}) \tag{29}$$

for some suitable function $\Psi : [0, 1]^d \mapsto [0, \infty)$ which has continuous partial derivatives. If the density exists, we denote it by $c(\mathbf{u}; \theta) = 1 + \theta\psi(\mathbf{u})$. For dimension $d = 2$, this copula has been considered by a number of authors starting with Farlie (1960). The Farlie-Gumbel-Morgenstern copula, as given in equation (28) for $\lambda = 0$, is of that particular form.

Suppose $T^{(1)}_n$ equals Blomqvist's $\hat{\beta}_n(\mathbf{u}, \mathbf{v})$, $T^{(2)}_n$ is Spearman's $\hat{\rho}_n$ and $T^{(3)}_n = \hat{\rho}'_n$. Efficiency considerations of Blomqvist's multivariate $\hat{\beta}_n(\mathbf{u}, \mathbf{v})$ for general $\mathbf{u}$ and $\mathbf{v}$ are interesting if emphasis is put on the analysis of the tail region of a copula. Under the copula model (29), the statistics $\sqrt{n}\{T^{(i)}_n - \mu_i(\theta)\},\ i = 1, 2, 3$, are asymptotically normal with standard deviation $\sigma_i(\theta)$. We test the hypothesis of independence

$$H_0 : \theta = 0 \tag{30}$$

against the one-sided alternative

$$H_1 : \theta > 0.$$

The test rejects H_0 if $\sqrt{n}\{T_n^{(i)} - \mu_i(0)\}/\sigma_i(0)$ exceeds $z_{1-\alpha},\ \alpha > 0$. Here, $z_{1-\alpha}$ is such that $\Phi(z_{1-\alpha}) = 1 - \alpha$ with standard normal distribution Φ. For a finite sample size n, it is difficult to calculate the power function of the tests. Thus, we investigate the asymptotic behavior of the power functions. Let $L_n^{(i)}(\theta)$ be the power function of test $T_n^{(i)}$. Since the asymptotic power function $\lim_{n\to\infty} L_n^{(i)}(\theta)$ is trivial for fixed θ, we consider the (local) asymptotic power function $\lim_{n\to\infty} L_n^{(i)}(\theta_n)$ with local parameter $\theta_n = c/\sqrt{n},\ c \geq 0$, if the limit exists. The next proposition establishes this limit for the test based on Blomqvist's $\hat{\beta}_n(\mathbf{u}, \mathbf{v})$.

Proposition 12 *Let $(\mathbf{X}_i)_{i=1,\ldots,n}$ be a sample from a d-dimensional distribution F with radially symmetric copula C specified in formula (29). Suppose the function ψ appearing in the density of C is bounded. Then, the (local) asymptotic power function of testing hypothesis (30) via $\hat{\beta}_n(\mathbf{u}, \mathbf{v})$ takes the form*

$$\lim_{n\to\infty} L_n^{(1)}(\theta_n) = 1 - \Phi[z_{1-\alpha} - c\, h_d(\mathbf{u}, \mathbf{v})\{\psi(\mathbf{u}) + \psi(\mathbf{1} - \mathbf{v})\}/\sigma_{\mathbf{u},\mathbf{v}}]$$

where $\theta_n = c/\sqrt{n}$ for arbitrary but fixed $c \geq 0$ and $\sigma_{\mathbf{u},\mathbf{v}}$ is given in formula (19).

Proof. Let P_n denote the probability measure under the null hypothesis ($\theta = 0$) and Q_n be the probability measure under the alternative ($\theta_n = c/\sqrt{n}$). The statistics $\hat{\beta}_n(\mathbf{u}, \mathbf{v}; \theta)$ will be indexed by θ. First, we show that the sequence $(\hat{\beta}_n(\mathbf{u}, \mathbf{v}; \theta_n))$ is locally uniformly asymptotically normal, i.e.

$$S_n(\theta_n) := \sqrt{n}\frac{\hat{\beta}_n(\mathbf{u}, \mathbf{v}; \theta_n) - \mu_1(\theta_n)}{\sigma_1(\theta_n)} \xrightarrow{d} N(0, 1).$$

In order to verify this, we utilize Le Cam's third lemma. We show that the vector $(S_n(0), \ln Q_n/P_n)$ is asymptotically normally distributed, where the second component denotes the logarithm of the likelihood ratio. In the following we derive a Taylor expansion of the logarithm of the likelihood ratio. The Fisher information for θ is given by

$$\begin{aligned} I_\theta &= \int_{\mathbb{R}^d} \left\{\frac{\dot{f}_\theta(\mathbf{x})}{f_\theta(\mathbf{x})}\right\}^2 f_\theta(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^d} \left\{\frac{\psi(F_1(x_1), \ldots, F_d(x_d))}{1 + \theta\psi(F_1(x_1), \ldots, F_d(x_d))}\right\}^2 dF(\mathbf{x}) \\ &= \int_{[0,1]^d} \left\{\frac{\psi(\mathbf{u})}{1 + \theta\psi(\mathbf{u})}\right\}^2 dC(\mathbf{u}) \end{aligned} \tag{31}$$

where f_θ and ψ, respectively, are the densities of F and Ψ, and $\dot{f}_\theta$ is the derivative with respect to θ. Moreover, F_i denotes the i-th marginal distribution function of F. According to the prerequisites, the Fisher information I_θ is well defined and continuous in $\theta \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Furthermore, ε can be chosen such that the map $\theta \mapsto \sqrt{1 + \theta\psi(\mathbf{u})}$ is differentiable at every $\mathbf{u}$. Thus, by Lemma 7.6 in Van der Vaart (1998), p.95, the map is differentiable in quadratic mean - see the last reference for the definition - which is a sufficient condition for the following expansion:

$$\begin{aligned} \ln\frac{Q_n}{P_n} &= \ln\prod_{j=1}^n \frac{f_{\theta_n}}{f_\theta}\Big(\mathbf{X}_j\Big)\Big|_{\theta=0} \\ &= \frac{c}{\sqrt{n}}\sum_{j=1}^n \frac{\dot{f}_\theta}{f_\theta}\Big(\mathbf{X}_j\Big)\Big|_{\theta=0} - \frac{1}{2}c^2 I_\theta\Big|_{\theta=0} + o_{P_n}(1). \end{aligned}$$

Hence, utilizing the empirical distribution function F_n of F, the vector

$$\Big[\sqrt{n}\{F_n(\mathbf{u}) - F(\mathbf{u})\}, \sqrt{n}\{\bar{F}_n(\mathbf{v}) - \bar{F}(\mathbf{v})\}, \ln Q_n/P_n\Big]$$

is jointly asymptotically normal by the multivariate central limit theorem and Slutsky's lemma. Mimicking the delta method utilized in the proof of Theorem 2, the vector $(S_n(0), \ln Q_n/P_n)$ is asymptotically bivariate normal. Thus, by Le Cam's third lemma, the sequence $S_n(\theta_n)$ converges - under the alternative $\theta_n = c/\sqrt{n}$ - to a standard normal distribution.

Further, $\mu(\theta) = \beta(\mathbf{u}, \mathbf{v}; \theta)$ is differentiable with respect to θ and the variance $\sigma^2_{\mathbf{u},\mathbf{v}}(\theta)$ is continuous at $\theta = 0$. The first assertion follows by the definition of Blomqvist's β and the specific form of the copula C defined in (29). The second assertion follows by formula (10) in Theorem 2 which implies that $\sigma^2_{\mathbf{u},\mathbf{v}}(\theta)$ is polynomial in θ. Under the null hypothesis, $\sqrt{n}\,\hat{\beta}_n(\mathbf{u}, \mathbf{v})$ is asymptotically normal with mean zero and variance $\sigma^2_{\mathbf{u},\mathbf{v}} = \sigma^2_{\mathbf{u},\mathbf{v}}(0)$ given in formula (19). Thus, the power function of the test that rejects the null hypothesis if $\sqrt{n}\,\hat{\beta}_n(\mathbf{u}, \mathbf{v})/\sigma_{\mathbf{u},\mathbf{v}}$ exceeds $z_{1-\alpha}$ takes the form

$$L_n^{(1)}(\theta_n) = 1 - \Phi\Big[\frac{\sigma_{\mathbf{u},\mathbf{v}} z_{1-\alpha} - \sqrt{n}\{\beta(\mathbf{u}, \mathbf{v}; \theta_n) - 0\}}{\sigma_{\mathbf{u},\mathbf{v}}(\theta_n)}\Big] + o(1).$$

Finally, observe that $\sqrt{n}\beta(\mathbf{u}, \mathbf{v}; \theta_n) = c\frac{d\beta(\mathbf{u},\mathbf{v};\theta)}{d\theta}\Big|_{\theta=0} + o(n) = c\, h_d(\mathbf{u}, \mathbf{v})\{\psi(\mathbf{u}) + \psi(\mathbf{1} - \mathbf{v})\} + o(n)$ for $\theta_n = c/\sqrt{n},\ c \geq 0$. $\square$

The following measure PE - called Pitman efficiency - is a recognized measure of comparison between two tests $T_n^{(k)}$ and $T_n^{(l)}$. We define

$$PE_{k,l} := \left(\frac{\mu'_k(0)/\sigma_k(0)}{\mu'_l(0)/\sigma_l(0)}\right)^2.$$

Under the conditions of Theorem 14.19 in Van der Vaart (1998), the measure $PE_{k,l}$ equals

$$PE_{k,l} = \lim_{\nu\to\infty} \frac{n_{\nu,l}}{n_{\nu,k}}$$

if the limit exists, where $n_{\nu,i}$ is the minimal number of observations such that $L^{(i)}_{n_{\nu,i}}(0) \leq \alpha$ and $L^{(i)}_{n_{\nu,i}}(\theta_\nu) \geq \gamma$ for fixed $\gamma \in (\alpha, 1)$. Note that the above limit is independent of the choice of $\alpha > 0$ and $\gamma \in (\alpha, 1)$. Similarly to (27), this quotient indicates the proportion of observations the sequence of estimators $(T_n^{(l)})$ needs in order to achieve the same (asymptotic) precision as $(T_n^{(k)})$ for the above hypothesis test. In our setting we derive

$$PE_{1,2} = \left[\frac{h_d(\mathbf{u}, \mathbf{v})\{\psi(\mathbf{u}) + \psi(\mathbf{1} - \mathbf{v})\}}{12\binom{d}{2}^{-1/2}\sigma_{\mathbf{u},\mathbf{v}} \int_{[0,1]^d} \sum_{k<j} w_k w_j \psi(\mathbf{w})\, d\mathbf{w}}\right]^2 \quad \text{and}$$

$$PE_{1,3} = \left[\frac{h_d(\mathbf{u}, \mathbf{v})\{\psi(\mathbf{u}) + \psi(\mathbf{1} - \mathbf{v})\}}{2^d\Big\{\left(\frac{4}{3}\right)^d - \frac{d}{3} - 1\Big\}^{-1/2}\sigma_{\mathbf{u},\mathbf{v}} \int_{[0,1]^d} \prod_{i=1}^d w_i \psi(\mathbf{w})\, d\mathbf{w}}\right]^2.$$

The (local) asymptotic power functions of the sample versions of Spearman's ρ, given by $\hat{\rho}_n$ and $\hat{\rho}'_n$, are derived in Lemmas 2 and 3 in Stepanova (2003).

Under the conditions of Proposition 12, we derive the following upper bound of the (local) asymptotic power function (cf. Theorem 15.4 in Van der Vaart (1998)) for testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$ using the above described testing procedure:

$$\limsup_{n\to\infty} L_n^{(1)}(\theta_n) \leq 1 - \Phi(z_{1-\alpha} - c\sqrt{I_0}), \quad c \geq 0,$$

where $I_0 = \int_{[0,1]^d} \{\psi(\mathbf{w})\}^2\, d\mathbf{w}$ is the Fisher information at $\theta = 0$ - see formula (31). Thus, the Pitman efficiency of Blomqvist's $\hat{\beta}_n(\mathbf{u}, \mathbf{v})$ with respect to the best test is given by

$$PE = h_d^2(\mathbf{u}, \mathbf{v})\{\psi(\mathbf{u}) + \psi(\mathbf{1} - \mathbf{v})\}^2 \Big/ \Big[\sigma^2_{\mathbf{u},\mathbf{v}} \int_{[0,1]^d} \{\psi(\mathbf{w})\}^2\, d\mathbf{w}\Big].$$

Obviously, this Pitman efficiency is higher if the probability mass of the copula C is concentrated around the points $\mathbf{u}$ or $\mathbf{v}$, which is also illustrated by the next example.

Example. Consider a bivariate copula C, as defined in (29), with a suitable function $\Psi(u_1, u_2) = g(u_1)h(u_2)$. Conditions such that C is a copula are given in Rodríguez-Lallena and Úbeda-Flores (2004). Then Blomqvist's $\beta = \beta(\mathbf{1/2}, \mathbf{1/2}) = 4\theta g(1/2)h(1/2)$ and Spearman's ρ is

$$\rho = 12\theta \int_0^1 g(u)\, du \int_0^1 h(u)\, du.$$

The asymptotic variance at $\theta = 0$ is equal to one for both statistics. Thus, the Pitman efficiency of Blomqvist's β with respect to Spearman's ρ is given by

$$PE = \Big[3 \int_0^1 g(u)/g(1/2)\, du \int_0^1 h(u)/h(1/2)\, du\Big]^{-2}.$$

For the Farlie-Gumbel-Morgenstern copula - cf. equation (28) - the $PE = 3/4$. However, for suitable functions g and h, the PE may become arbitrarily large.

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Appendix

We give some alternative representations of Blomqvist's multivariate $\beta = \beta(\mathbf{1/2}, \mathbf{1/2})$ as defined in formula (4). Representations for general Blomqvist's multivariate $\beta(\mathbf{u}, \mathbf{v})$ can be derived similarly. First, β can be expressed (cf. Úbeda-Flores 2005, Theorem 3.1) via

$$\beta = h_d \left\{ P(\mathbf{X} < \tilde{\mathbf{x}} \text{ or } \mathbf{X} > \tilde{\mathbf{x}}) - 2^{1-d} \right\} = h_d \left\{ C\left(\mathbf{1/2}\right) + C_S\left(\mathbf{1/2}\right) - 2^{1-d} \right\},$$

where $\tilde{\mathbf{x}}$ denotes the vector of medians corresponding to the univariate marginal distributions and C_S refers to the survival copula, which was defined by $C_S(\mathbf{u}) = P(\mathbf{1} - \mathbf{U} \leq \mathbf{u})$.

For some subset $I \subset \{1, ..., d\}$ we define $B_I = \{\mathbf{u} \in [0,1]^d \mid u_i \leq \frac{1}{2} \text{ for } i \in I \text{ and } u_i > \frac{1}{2} \text{ for } i \notin I\}$. Then another representation of β is given by

$$\beta = h_d \Big[\sum_{k=1}^{d-1} \sum_{|I|=k} \left\{ 2^{-d} - P_C(B_I) \right\} - 2^{1-d} \Big], \quad \text{since} \quad \sum_{k=0}^{d} \sum_{|I|=k} P_C\left(B_I\right) = 1,$$

where P_C denotes the probability measure induced by the copula C.

Assume C is radially symmetric at $\mathbf{1/2}$, i.e. $C\left(\mathbf{1/2}\right) = \bar{C}\left(\mathbf{1/2}\right)$, which is equivalent to $P\{X_i \leq F_i^{-1}(1/2) \text{ for all } i = 1, ..., d\} = P\{X_i \geq F_i^{-1}(1/2) \text{ for all } i = 1, ..., d\}$. This symmetry holds if C is radially symmetric, i.e. $C \equiv C_S$, but is much weaker. Then obviously

$$\beta \quad = \quad h_d \left\{ 2C\left(\mathbf{1/2}\right) - 2^{1-d} \right\}.$$

In this case there is an interesting relationship between Blomqvist's β in d dimensions and those of lower dimensions $d' \leq d$. We use the notation β_I for Blomqvist's β of those variables X_i where $i \in I \subset \{1, ..., d\}$. If d is odd, we have

$$\beta_{\{1,...,d\}} = \sum_{k=2}^{d-1} (-1)^k \frac{1/2 - 2^{-k}}{1 - 2^{1-d}} \sum_{\substack{I \subset \{1,...,d\} \\ |I|=k}} \beta_I = \sum_{k=2}^{d-1} \sum_{\substack{I \subset \{1,...,d\} \\ |I|=k}} b_{d,k} \beta_I$$

with weights $b_{d,k} = (-1)^k(1/2 - 2^{-k})/(1 - 2^{1-d})$. Thus, the d-dimensional β can be written as a weighted sum of the d'-dimensional β's with $2 \leq d' \leq d-1$ since $\sum_{k=2}^{d-1} \binom{d}{k} b_{d,k} = 1$. This follows by the inclusion-exclusion principle which implies

$$C(\mathbf{1/2}) = \sum_{k=0}^{d} (-1)^k \sum_{\substack{I \subset \{1,\ldots,d\} \\ |I|=k}} C\left((\mathbf{1/2})^I\right) \quad \text{with } (\mathbf{1/2})_i^I = \begin{cases} \frac{1}{2} & , \quad i \in I, \\ 1 & , \quad i \notin I. \end{cases}$$

Hence, if d is odd, we have

$$\begin{aligned} 2\left(\frac{1}{2} - \left(\frac{1}{2}\right)^d\right)\beta_{\{1,\ldots,d\}} &= \sum_{k=2}^{d-1}(-1)^k \sum_{\substack{I \subset \{1,\ldots,d\} \\ |I|=k}} \left(\frac{1}{2} - \left(\frac{1}{2}\right)^k\right)\beta_I + \left(\frac{1}{2}\right)^k \\ &= \underbrace{\sum_{k=0}^{d}\binom{d}{k}(-1)^k\left(\frac{1}{2}\right)^k}_{=(\frac{1}{2})^d} + \sum_{k=2}^{d}(-1)^k \sum_{\substack{I \subset \{1,\ldots,d\} \\ |I|=k}} \left(\frac{1}{2} - \left(\frac{1}{2}\right)^k\right)\beta_I. \end{aligned}$$

Exemplarily, for $d = 3$ we have

$$\beta_{\{1,2,3\}} = \frac{1}{3}\beta_{\{1,2\}} + \frac{1}{3}\beta_{\{1,3\}} + \frac{1}{3}\beta_{\{2,3\}}.$$

If d is even, $\beta_{\{1,\ldots,d\}}$ cannot be expressed as a linear combination of β_I for $2 \leq |I| \leq d-1$. This is the case as the inclusion-exclusion principle implies

$$\sum_{k=2}^{d-1}(-1)^k \sum_{\substack{I \subset \{1,\ldots,d\} \\ |I|=k}} \left(\frac{1}{2} - \left(\frac{1}{2}\right)^k\right)\beta_I = 0.$$