F.Schmid and R. Schmidt, 2007, Multivariate Conditional Versions of Spearman's Rho and Related Measures of Tail Dependence. *The Journal of Multivariate Analysis,* **98**, Elsevier, 1123–1140.

Multivariate Conditional Versions of Spearman's Rho and Related Measures of Tail Dependence

Friedrich Schmid[1] and Rafael Schmidt[12]

Abstract

A new family of conditional dependence measures based on Spearman's rho is introduced. The corresponding multidimensional versions are established. Asymptotic distributional results are derived for related estimators which are based on the empirical copula. Particular emphasis is placed on a new type of multidimensional tail-dependence measure and its relationship to other measures of tail dependence is shown. Multivariate tail dependence describes the limiting amount of dependence in the vertices of the copula's domain.

AMS 2000 subject class.: Primary 62H20, 60F05, 62G20, Secondary 60G15, 60G70.

Key words: Measure of dependence, Copula, Spearman's rho, Tail dependence, Empirical copula, Weak convergence.

1 Introduction

Conditional versions of dependence measures are of interest both in theory and practice. For example, a common conditional version of Pearson's correlation coefficient r of a bivariate random vector $\mathbf{X} = (X_1, X_2)'$ is defined via $r_A(X_1, X_2) := r(X_1, X_2 \mid \mathbf{X} \in A)$ for some (measurable) set $A \subset \mathbb{R}^2$, i.e., Pearson's correlation coefficient is derived from the conditional distribution function $P(\mathbf{X} \leq \mathbf{x} \mid \mathbf{X} \in A)$. In particular in financial engineering, this conditional dependence measure is frequently used in order to investigate the effects of contagion between financial markets, see Campbell, Koedijk, and Kofman (2002) or Forbes and Rigobon (2002). Unfortunately, Pearson's correlation coefficient is often an unsuitable dependence measure since, firstly, it measures linear dependence, secondly, it is not invariant to a change of the univariate margins, and thirdly, it is very sensitive to outliers. The related pitfalls have been pointed out by a large number of authors, we mention Embrechts, McNeil, and Straumann (2002). Further, this dependence measure cannot be expressed via the copula. However, it is precisely the copula of a random vector which captures those properties of the joint distribution which are invariant under (strictly increasing) transformations of the univariate margins - so-called scale invariance - see Schweizer and Wolff (1981). It is, thus, natural to consider versions of alternative dependence measures which are based on the distribution's copula. Possible alternatives are Spearman's rho, Kendall's tau or Blomqvist's beta. In this paper we will concentrate on Spearman's rho which is best-known in economic and social statistics. We think that conditional versions of Kendall's tau and Blomqvist's beta are similarly interesting to consider. They will be the focus of subsequent work.

[1]Department of Economic and Social Statistics, University of Cologne, Germany. rafael.schmidt@uni-koeln.de

[2]Department of Statistics, London School of Economics, UK. The author gratefully acknowledges financial support by the Deutsche Forschungsgemeinschaft (DFG).

We can think of various conditional versions of Spearman's rho. One possible version, which is motivated by the conditional version of Pearson's correlation coefficient, is defined as

$$\rho_{S,A}(X_1, X_2) := \rho_S(X_1, X_2 \mid \mathbf{X} \in A), \quad A \subset \mathbb{R}^2, \tag{1}$$

where ρ_S corresponds to Spearman's rho. The multidimensional generalization of this conditional dependence measure, however, is not straightforward and its analytical and statistical tractability is rather limited, as we will see. Another class of conditional versions of Spearman's rho utilizes weighting functions which weight the different parts of the copula. This approach is quite common in probability and statistics, e.g., in goodness-of-fit statistics which emphasize the tail region of the distribution. The conditional dependence measures considered in this paper will be of this particular type. The corresponding family of weighting functions is quite general, thus providing us with a large variety of dependence measures. The multidimensional extension of Spearman's rho we use has also been considered by Ruymgaart and van Zuijlen (1978), Wolff (1980), Joe (1990), and Nelsen (1996). Special emphasis is placed on conditional dependence measures which measure the amount of dependence in the lower tail of multivariate distributions. In particular the limiting behavior of these dependence measures, if we tend to the lower end point of each univariate marginal distribution function, will be of interest. In the bivariate setting this limiting dependence is commonly measured via the so-called tail-dependence coefficient (Sibuya 1960). We propose a new multivariate measure of tail dependence and establish its relationship to the tail-dependence coefficient.

For the statistical inference, two cases may be distinguished. In the first case, the marginal distribution functions are assumed to be known which is, however, a nonrelevant case for most applications. We therefore concentrate on the second case assuming unknown marginal distribution functions. This leads to a consideration of the empirical copula. By means of a weak convergence result for multivariate empirical copulae - given in Rüschendorf (1976), Stute (1984), Fermanian, Radulović, and Wegkamp (2004) or Tsukahara (2005) - we establish weak convergence of the respective estimators which is uniform with respect to the weighting function. Further, we provide a bootstrap method for estimating the limiting covariance structure. The asymptotic behavior of the tail-dependence measure is derived from so-called empirical tail copulae.

Section 2 provides the necessary notation and definitions. Thereafter, Section 3 introduces the family of multivariate conditional versions of Spearman's rho. We start with two generalizations of Spearman's rho to d dimensions in Section 3.1. Afterwards, we define the conditional dependence measures by weighting different parts of the copula via suitable weighting functions. Section 3.3 presents a new type of tail-dependence measure and investigates its relationship with the well-known tail-dependence coefficient. The statistical estimation of these dependence measures is addressed in Section 4.

2 Notation and Definitions

Throughout the paper we write bold letters for vectors, e.g., $\mathbf{x} := (x_1, \ldots, x_d)'$. Inequalities $\mathbf{x} \leq \mathbf{y}$ are understood componentwise, i.e, $x_i \leq y_i$ for all $i = 1, \ldots, d$. The indicator function on a set A is denoted by $\mathbf{1}_A$. The set $[a, b]^d$, $a < b$, refers to the d-dimensional cartesian product $[a, b] \times \cdots \times [a, b] \subset \mathbb{R}^d$ and $R_+^d = [0, \infty)^d$. The positive part of x is written as $x^+ = \max(x, 0)$. Let $\ell^\infty(T)$ denote the space of all uniformly bounded real-valued functions on some set T.

Let $X_1, X_2, \ldots, X_d$ be $d \geq 2$ random variables with joint distribution function

$$F(\mathbf{x}) = P(X_1 \leq x_1, \ldots, X_d \leq x_d), \quad \mathbf{x} = (x_1, \ldots, x_d)' \in \mathbb{R}^d,$$

and marginal distribution functions $F_{X_i}(x_i) = P(X_i \leq x_i)$ for $x_i \in \mathbb{R}$ and $i = 1, \ldots, d$. We assume that the F_{X_i} are continuous functions. Thus, according to Sklar's theorem (Sklar 1959), there exists a unique copula $C : [0, 1]^d \to [0, 1]$ such that

$$F(\mathbf{x}) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)), \quad \text{for all } \mathbf{x} \in \mathbb{R}^d.$$

The copula C is the joint distribution function of the random variables $U_i = F_{X_i}(X_i)$, $i = 1, \ldots, d$. Moreover, $C(\mathbf{u}) = F(F_{X_1}^{-1}(u_1), \ldots, F_{X_d}^{-1}(u_d))$ for all $\mathbf{u} \in [0,1]^d$. The generalized inverse function G^{-1} is defined via $G^{-1}(u) := \inf\{x \in \mathbb{R} \cup \{\infty\} \mid G(x) \geq u\}$ for all $u \in (0,1]$ and $G^{-1}(0) := \sup\{x \in \mathbb{R} \cup \{-\infty\} \mid G(x) = 0\}$. A detailed treatment of copulae is given in Nelsen (1999) and Joe (1997).

Every copula C is bounded in the following sense:

$$W(\mathbf{u}) := \max\{u_1 + \ldots + u_d - (d-1), 0\} \\ \leq C(\mathbf{u}) \leq \min\{u_1, \ldots, u_d\} =: M(\mathbf{u}) \quad \text{for all } \mathbf{u} \in [0,1]^d,$$

where M and W are called the upper and lower *Fréchet-Hoeffding bounds*, respectively. The upper bound M is a copula itself and is also known as the comonotonic copula. It represents the copula of $X_1, \ldots, X_d$ if $F_{X_1}(X_1) = \cdots = F_{X_d}(X_d)$ with probability one, thus, if there exists an almost surely strictly increasing functional relationship between X_i and X_j $(i \neq j)$. By contrast, the lower bound W is a copula only for dimension $d = 2$. Another important copula is the independence copula

$$\Pi(\mathbf{u}) := \prod_{i=1}^{d} u_i, \quad \mathbf{u} \in [0,1]^d,$$

describing the dependence structure of stochastically independent random variables $X_1, \ldots, X_d$.

3 Multivariate Conditional Versions of Spearman's Rho

The present section introduces multivariate conditional versions of Spearman's rho. Various examples are given. A new measure of multivariate tail dependence is presented and its relationship to a well-known bivariate measure of tail dependence is examined.

3.1 Generalization of Spearman's Rho to d Dimensions

In order to motivate the multivariate conditional versions of Spearman's rho, we first focus on possible generalizations of Spearman's rho to higher dimensions. Recall that Spearman's rho ρ_S (Spearman 1904) of a two-dimensional random vector $\mathbf{X} = (X_1, X_2)'$ with distribution function F, univariate marginal distribution functions F_{X_1}, F_{X_2}, and copula C is defined by

$$\begin{aligned} \rho_S &= \frac{cov\left(F_{X_1}(X_1), F_{X_2}(X_2)\right)}{\sqrt{var\left(F_{X_1}(X_1)\right)}\sqrt{var\left(F_{X_2}(X_2)\right)}} = \frac{cov\left(U_1, U_2\right)}{\sqrt{var\left(U_1\right)}\sqrt{var\left(U_2\right)}} \\ &= \frac{\int_0^1\int_0^1 uv dC(u,v) - \left(\frac{1}{2}\right)^2}{\sqrt{\frac{1}{12}}\sqrt{\frac{1}{12}}} = 12\int_0^1\int_0^1 C(u,v)\,dudv - 3, \end{aligned}$$

where $(U_1, U_2)'$ are distributed with copula C. The following alternative representation is readily verified and plays a central role in the forthcoming definitions of conditional versions of Spearman's rho

$$\rho_S = \frac{\int_0^1\int_0^1 C(u,v)\,dudv - \int_0^1\int_0^1 uv\ dudv}{\int_0^1\int_0^1 \min\{u,v\}\,dudv - \int_0^1\int_0^1 uv\ dudv} = \frac{\int_0^1\int_0^1 C(u,v)\ dudv - \int_0^1\int_0^1 \Pi(u,v)\ dudv}{\int_0^1\int_0^1 M(u,v)\ dudv - \int_0^1\int_0^1 \Pi(u,v)\ dudv}. \tag{2}$$

Thus, ρ_S can be interpreted as the normalized average distance between the copula C and the independence copula $\Pi(u,v) = uv$.

Several multidimensional extensions of Spearman's rho have been discussed in the literature, we mention Ruymgaart and van Zuijlen (1978), Wolff (1980), Joe (1990), and Nelsen (1996). For example, Wolff (1980) introduces the following straightforward generalization of Spearman's rho ρ_S as given by representation (2),

$$\rho_d = \frac{\int\limits_{[0,1]^d} C(\mathbf{u})\, d\mathbf{u} - \int\limits_{[0,1]^d} \Pi(\mathbf{u})\, d\mathbf{u}}{\int\limits_{[0,1]^d} M(\mathbf{u})\, d\mathbf{u} - \int\limits_{[0,1]^d} \Pi(\mathbf{u})\, d\mathbf{u}} = \frac{d+1}{2^d-(d+1)} \Big\{ 2^d \int\limits_{[0,1]^d} C(\mathbf{u})\, d\mathbf{u} - 1 \Big\}. \qquad (3)$$

Further, Ruymgaart and van Zuijlen (1978) address the estimation of the alternative measure

$$\tilde{\rho}_d = \frac{\int\limits_{[0,1]^d} \Pi(\mathbf{u})\, dC(\mathbf{u}) - \int\limits_{[0,1]^d} \Pi(\mathbf{u})\, d\mathbf{u}}{\int\limits_{[0,1]^d} M(\mathbf{u})\, d\mathbf{u} - \int\limits_{[0,1]^d} \Pi(\mathbf{u})\, d\mathbf{u}} = \frac{d+1}{2^d-(d+1)} \Big\{ 2^d \int\limits_{[0,1]^d} \Pi(\mathbf{u})\, dC(\mathbf{u}) - 1 \Big\}.$$

Both generalizations ρ_d and $\tilde{\rho}_d$ coincide with Spearman's rho if $d = 2$. In particular, Nelsen (1996) shows that ρ_d can be derived from the concept of average *lower* orthant dependence, whereas $\tilde{\rho}_d$ represents a measure calculated from the concept of average *upper* orthant dependence. In general, $\rho_d \neq \tilde{\rho}_d$ for dimension $d \geq 3$, except for the case where the copula C is radially symmetric, i.e., $C(\mathbf{u}) = P(\mathbf{U} \leq \mathbf{u}) = P(\mathbf{U} > \mathbf{1} - \mathbf{u}) = \bar{C}(\mathbf{1} - \mathbf{u})$. Both types of multivariate generalizations of Spearman's rho are interesting. However, we elaborate only on the better known measure ρ_d, though most of the analytical results of the next section can be immediately transferred to $\tilde{\rho}_d$. The asymptotical behavior of the estimators for $\tilde{\rho}_d$, which are based on the empirical copula, are harder to analyze. We will drop the index d for notational convenience.

3.2 Conditional Versions of Spearman's Rho

The following definition of the multivariate conditional version of Spearman's rho is motivated by Formula (3):

$$\rho(g) := \frac{\int\limits_{[0,1]^d} C(\mathbf{u})g(\mathbf{u})\, d\mathbf{u} - \int\limits_{[0,1]^d} \Pi(\mathbf{u})g(\mathbf{u})\, d\mathbf{u}}{\int\limits_{[0,1]^d} M(\mathbf{u})g(\mathbf{u})\, d\mathbf{u} - \int\limits_{[0,1]^d} \Pi(\mathbf{u})g(\mathbf{u})\, d\mathbf{u}} \qquad (4)$$

for some measurable function $g \geq 0$ such that the integrals exist. The function g will be called weighting function because it weights specific parts of the copula which are of interest, e.g, the lower or upper tails of the copula. Later, we will impose further conditions on g in order to derive interesting asymptotic results for related statistics.

In order to obtain concrete examples and to define a new multivariate concept of tail dependence, we will consider weighting functions g of the form $g(\mathbf{u}) = \mathbf{1}_{[0,p]^d}(\mathbf{u}),\ 0 < p \leq 1$, in more detail. Note that these weighting functions refer to the lower part of the copula C. The resulting d-dimensional conditional version of Spearman's rho for $0 < p \leq 1$ is defined by

$$\rho(p) := \frac{\int_{[0,p]^d} C(\mathbf{u})d\mathbf{u} - \int_{[0,p]^d} \Pi(\mathbf{u})d\mathbf{u}}{\int_{[0,p]^d} M(\mathbf{u})d\mathbf{u} - \int_{[0,p]^d} \Pi(\mathbf{u})d\mathbf{u}} = \frac{\int_{[0,p]^d} C(\mathbf{u})d\mathbf{u} - (\frac{p^2}{2})^d}{\frac{p^{d+1}}{d+1} - (\frac{p^2}{2})^d}. \qquad (5)$$

The dependence measure $\rho(p)$ preserves the ordering of concordance, i.e., if $C(\mathbf{u}) \leq C'(\mathbf{u})$ and $\bar{C}(\mathbf{u}) \leq \bar{C}'(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^d$, then $\rho_C(p) \leq \rho_{C'}(p)$ for all $0 < p \leq 1$. This preservation of the concordance order holds also for $\rho(g)$. In other words, $\rho(g)$ is a measure of concordance.

For $p = 1$, $\rho(p)$ coincides with the (unconditional) multivariate version of Spearman's rho, as given in Formula (3). Note that

$$M(\mathbf{u}) \geq C(\mathbf{u}) \geq W(\mathbf{u}) \quad \text{implies} \quad 1 \geq \rho(p) \geq -\Big(\frac{p^2}{2}\Big)^d \Big/ \Big\{ \frac{p^{d+1}}{d+1} - \Big(\frac{p^2}{2}\Big)^d \Big\}.$$

It becomes clear that the lower bound for $\rho(p)$ tends quickly to zero with increasing dimension. Below, we illustrate $\rho(p)$ with some examples. We remark that the limit of $\rho(p)$ may not exist if

p tends to zero. This issue will be addressed later when we discuss tail dependence. If the limit exists, we include $p = 0$ in the domain of $\rho(p)$.

Before we proceed, let us discuss the alternative conditional measure of Spearman's rho given in the Introduction. A multivariate version of Formula (1) can be obtained via the generalization (3) of Spearman's rho. This conditional version of Spearman's rho would involve the copula of the conditional joint distribution function $P(\mathbf{X} \leq \mathbf{x} \mid \mathbf{X} \in A)$. Even for $A = (-\infty, F_{X_1}^{-1}(p)] \times \cdots \times (-\infty, F_{X_d}^{-1}(p)]$, this copula takes the rather complicated form:

$$\frac{C(h_{1p}^{-1}(u_1), \ldots, h_{dp}^{-1}(u_d))}{C(\mathbf{p})} \quad \text{with functions} \quad h_{ip}(x_i) = \frac{C(p, \ldots, p, x_i, p, \ldots, p)}{C(\mathbf{p})},\ i = 1, \ldots, d,$$

and $\mathbf{p} = (p, \ldots, p)' \in [0,1]^d$. Certainly, this version would be interesting to investigate, too, although its analytics and the nonparametrical statistical inference, as discussed in Section 4 later, are difficult. By contrast, the analytical structure and the nonparametric estimation of the conditional measures in Formula (4) are more convenient and intuitive, as we will see.

Examples. i) The bivariate Farlie-Morgenstern copula $C(u, v; \theta) = uv + \theta uv(1-u)(1-v), \theta \in [-1, 1]$ yields

$$\rho(p) = \theta \cdot \frac{p^3/9 - p^2/3 + p/4}{1/3 - p/4}, \quad 0 \leq p \leq 1.$$

ii) For the family of Fréchet copulae

$$C(u, v; \theta_1, \theta_2) = \theta_1 W(u, v) + (1 - \theta_1 - \theta_2)\Pi(u, v) + \theta_2 M(u, v)$$

with $0 \leq \theta_1, \theta_2 \leq 1$ and $\theta_1 + \theta_2 \leq 1$ we obtain

$$\rho(p) = \begin{cases} \theta_2 - \theta_1 \cdot 3p^4/(4p^3 - 3p^4), & 0 \leq p \leq \frac{1}{2}, \\ \theta_2 - \theta_1 \cdot \dfrac{3p^4 - 16p^3 + 24p^2 - 12p + 2}{4p^3 - 3p^4}, & \frac{1}{2} < p \leq 1. \end{cases}$$

For $p = 1$, we obtain the (unconditional) Spearman's $\rho = \rho(1) = \theta_2 - \theta_1$.

iii) For the 3-dimensional Cuadras-Augé copula

$$C(u, v, w; \theta) = [\min(u, v, w)]^\theta (uvw)^{1-\theta}, \quad 0 \leq \theta \leq 1,$$

we have

$$\rho(p) = \left\{ \frac{3p^{6-2\theta}}{2(4-\theta)(3-\theta)} - p^6/8 \right\} \Big/ \left(p^4/4 - p^6/8 \right) \to \begin{cases} 1 & \text{if} \quad \theta = 1, \\ 0 & \text{if} \quad \theta \leq 0 < 1, \end{cases} \quad \text{as } p \to 0.$$

The explicit computation of the integral $\int C(\mathbf{u}) d\mathbf{u}$ is usually difficult for high-dimensional copulae. Hence, the next proposition establishes a useful result regarding the simulation of $\rho(p)$ for a given joint distribution or copula. The proof utilizes representation (5).

Proposition 1 *Let $\mathbf{X}$ be a d-dimensional random vector with copula C and univariate marginal distribution functions $F_{X_i}, i = 1, \ldots, d$. Let $Z_1, \ldots, Z_d$ be independent and uniformly distributed random variables on the interval $[0, p]$. Then*

$$\rho(p) \cdot \left\{ \frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2} \right)^d \right\} + \left(\frac{p^2}{2} \right)^d = P\{X_1 \leq F_{X_1}^{-1}(Z_1), \ldots, X_d \leq F_{X_d}^{-1}(Z_d)\} = E\{C(Z_1, \ldots, Z_d)\}.$$

The corresponding pseudo-simulation algorithm is given by:

(1) Generate $n \cdot d$ random numbers $z_{ij},\ i = 1, \ldots, d$ and $j = 1, \ldots, n$, which are independent and uniformly distributed on the interval $[0, p]$.

(2) Generate n random numbers $(x_{1j}, x_{2j}, \ldots, x_{dj})'$ with distribution function $F_{\mathbf{X}}$.

(3) Count the number k of $j = 1, \ldots, n$, where $x_{ij} \leq F_{X_i}^{-1}(z_{ij})$ for all $i = 1, \ldots, d$.

(4) Set $\rho_n(p) = \{\frac{k}{n} - (\frac{p^2}{2})^d\} / \{\frac{p^{d+1}}{d+1} - (\frac{p^2}{2})^d\}$.

3.3 A new Measure of Multivariate Tail Dependence

The concept of tail dependence helps to analyze and to model dependencies between extreme events. For example in finance, tail dependent distributions or copulae are frequently used in order to model the possible dependencies between large negative asset-returns or portfolio losses. More precisely, the (lower) tail-dependence coefficient λ_L between two random variables X_1 and X_2 with copula C is defined as

$$\lambda_L := \lim_{p \downarrow 0} P\{X_1 \le F_{X_1}^{-1}(p) \mid X_2 \le F_{X_2}^{-1}(p)\} = \lim_{p \downarrow 0} \frac{C(p,p)}{p}, \tag{6}$$

in case the limit exists. If $\lambda_L > 0$, we say that $\mathbf{X} = (X_1, X_2)'$ is tail dependent, otherwise $\mathbf{X}$ is tail independent. This dependence measure was introduced by Sibuya (1960) and plays a role in bivariate extreme value theory. For the independence copula $\Pi(u,v)$ we have $\lambda_L = 0$ (tail independence) and for the comonotonic copula $M(u,v)$ we have $\lambda_L = 1$ (tail dependence). Note that the tail-dependence coefficient λ_L is a copula-based dependence measure.

Unfortunately, the tail-dependence coefficient λ_L has some drawbacks. For example, it evaluates the copula C solely on its diagonal section, i.e., $C(p,p)$, $p \in [0,1]$. In other words, the limiting behavior, as defined in Formula (6), may be very different if we tend to the copula's lower left corner on a different route than on the main diagonal, e.g., if we consider $\lim_{p\downarrow 0} C(p, p/2)/p$. Regarding this drawback and other pitfalls, the reader may consult Schlather (2001), Abdous, Fougères, and Ghoudi (2005), and Frahm, Junker, and Schmidt (2005).

This motivates us to introduce the following multivariate measure of (lower) tail dependence which arises from the multivariate conditional version of Spearman's rho, as defined in Formula (5). As already mentioned in the introduction, similar measures can be derived from Kendall's tau or Blomqvist's beta. We define:

$$\rho_L := \lim_{p \downarrow 0} \rho(p) = \lim_{p \downarrow 0} \frac{d+1}{p^{d+1}} \int\limits_{[0,p]^d} C(\mathbf{u}) d\mathbf{u}, \tag{7}$$

in case the limit exists. Obviously $0 \le \rho_L \le 1$. Further, the comonotonic copula M implies $\rho_L = 1$ and the independence copula Π yields $\rho_L = 0$. Moreover, ρ_L preserves the concordance ordering.

The tail-dependence measure ρ_L can be written as the average of so-called tail copulae. A d-dimensional (*lower*) *tail copula* (Schmidt and Stadtmüller 2006) is defined by

$$\Lambda_L(\mathbf{x}) := \lim_{p \downarrow 0} C(p \cdot \mathbf{x})/p \quad \text{for} \quad \mathbf{x} \in [0,\infty]^d \backslash \{\infty\} \tag{8}$$

if the limit exists. The appropriate extension of the copula to this domain should be obvious. Hence, $\rho_L = \lim_{p\downarrow 0}(d+1) \int_{[0,1]^d} C(p \cdot \mathbf{u})/p \, d\mathbf{u} = (d+1) \int_{[0,1]^d} \Lambda_L(\mathbf{u}) \, d\mathbf{u}$.

Examples. i) The Farlie-Morgenstern copula yields $\rho_L = 0$.

ii) For the family of Fréchet copulae we obtain

$$\rho_L = \lim_{p \downarrow 0} \theta_2 - \theta_1 \cdot 3p^4/(4p^3 - 3p^4) = \theta_2.$$

An interesting feature of Fréchet copulae is that in case $\theta_1 = \theta_2$ we have Spearman's $\rho_S = \rho(1) = 0$, but $\rho_L = \theta_2$. Thus, the dependence in the lower tail of the copula might be large although Spearman's rho is zero. If $\theta_1 = \theta_2 = 1/2$, then $\rho_L = 0.5$, which is the maximal value of ρ_L for Fréchet copulae. The next proposition states the existence of copulae with a tail-dependence value $\rho_L = 1$, but with Spearman's $\rho_S = \rho(1)$ close to zero.

Proposition 2 *For any* $\varepsilon > 0$, *there exists a bivariate copula* C_ε *such that*

$$\rho_L = \lambda_L = 1 \quad \textit{and} \quad |\rho(p)| < \varepsilon \quad \textit{for all}\, p \in (\varepsilon, 1].$$

In particular, the inequality holds also for Spearman's ρ_S.

Proof. According to Theorem 3.2.1 in Nelsen (1999), for any $\varepsilon' > 0$ there exists a copula $C_{\varepsilon'}$ which is a shuffle of the comonotonic copula M such that $\sup_{u,v\in[0,1]} |C_{\varepsilon'}(u,v) - \Pi(u,v)| < \varepsilon'$. A shuffle is a copula whose support is a collection of line segments with slope $+1$ and -1. In particular, we may construct the shuffle $C_{\varepsilon'}$ in such a way that $C_{\varepsilon'}$ puts probability mass $1/n > 0$ on the sub-square $[0, 1/n]^2$ for each $n \in \mathbb{N}$ and $n \geq (4/\varepsilon')^2$. Thus, $\lambda_L = \lim_{n\to\infty} n \cdot C_{\varepsilon'}(1/n, 1/n) = 1$ for any $\varepsilon' > 0$. The fact that $\lambda_L = 1$ is equivalent to $\rho_L = 1$ is proven in Proposition 3. Set $\varepsilon' = \varepsilon/9 > 0$. Then

$$|\rho(p)| \leq \Big(\frac{p^3}{3} - \frac{p^4}{4}\Big) \int_{[0,p]^2} |C_{\varepsilon'}(u,v) - \Pi(u,v)| \, dudv \leq \Big(\frac{p^3}{3} - \frac{p^4}{4}\Big)\varepsilon' \leq \varepsilon. \qquad \square$$

The copula C_ε in Proposition 2, as constructed in the proof, shows that the tail-dependence measures λ_L and ρ_L can yield awkward results. In fact, we constructed a copula which is very close to the independence copula Π, but has $\lambda_L = \rho_L = 1$. By contrast, the dependence measure $\rho(p)$ is very small for nearly all $p \in (0, 1]$, as one would expect. This illustrates an advantage of the tail-dependence measure $\rho(p)$ for small p in this situation.

The next proposition establishes some results on how the tail-dependence measures ρ_L and λ_L interrelate with each other in the bivariate setting.

Proposition 3 *Let C be a bivariate copula with tail copula Λ_L. Then the following inequalities hold*

$$\lambda_L \leq \rho_L \leq \min\{1, 2\lambda_L\}. \tag{9}$$

Further, $\rho_L = 1 \Leftrightarrow \lambda_L = 1$ and $\rho_L = 0 \Leftrightarrow \lambda_L = 0$. Moreover,

$$2\lambda_L - \rho_L \geq \lim_{p\downarrow 0} \frac{1}{p}\Big\{ \int_0^p u \, dC_{U|V}(u \mid p) + \int_0^p v \, dC_{V|U}(v \mid p)\Big\}, \tag{10}$$

where $C_{U|V}$ and $C_{V|U}$ are the conditional copulae. If, in addition, C possesses continuous partial derivatives and the following limit exists, then

$$\rho_L - \lambda_L = \frac{1}{2} \lim_{p\downarrow 0} \int_0^1 \Big\{ \frac{\partial}{\partial y} C(pu, y) \mid_{y=p} + \frac{\partial}{\partial x} C(x, pu) \mid_{x=p} \Big\} \, du \tag{11}$$

Proof. According to Theorem 2.2.4 in Nelsen (1999), each copula C is uniformly continuous on its domain. Thus, if the limit ρ_L exists, l'Hospital's and the monotonicity of C imply

$$\rho_L = \lim_{p\downarrow 0} \frac{3}{p^3} \int_{[0,p]^2} C(u,v) \, dudv = \lim_{p\downarrow 0} \int_0^1 \frac{C(pu,p)}{p} \, du + \lim_{p\downarrow 0} \int_0^1 \frac{C(p,pv)}{p} \, dv \leq 2\lambda_L. \tag{12}$$

Inequality (10) follows now via partial integration, $\int_0^p C(u,p) \, du = p \cdot C(p,p) - \int_0^p u \, dC(u,p)$, and the fact that each copula has uniformly distributed margins. Assume now that the copula C has continuous partial derivatives and the limit in Equation (11) exists. Another application of l'Hospital's rule to this equation yields Equation (11). In this case, the inequality $\lambda_L \leq \rho_L$ follows from the fact that the above partial derivatives are greater than zero, since C is a distribution function. However, the partial derivatives only exist almost surely for arbitrary C. For general copulae C, we utilize Fatou's lemma to show that

$$\rho_L \geq \int_{[0,1]^2} \liminf_{p\downarrow 0} C(pu,pv)/p \, dudv \geq \liminf_{p\downarrow 0} \frac{C(p,p)}{p} \int_{[0,1]^2} \min(u,v) \, dudv = \lambda_L.$$

The latter inequality follows from the homogeneity of $u \mapsto \liminf_{p\downarrow 0} C(pu,pu)/p$ and the monotonicity of C. Further, the equivalence $\rho_L = 0 \Leftrightarrow \lambda_L = 0$ and $\lambda_L = 1 \Rightarrow \rho_L = 1$ is immediately given by Inequality (9). It remains to show that $\rho_L = 1$ implies $\lambda_L = 1$. Let Λ_L be the bivariate (lower) tail copula, as defined in Formula (8). For notational convenience, we drop the subscript L. Dominated convergence implies that $\rho_L = 1 = 3\int_{[0,1]^2} \Lambda(u,v) \, duv$. Assume that $\lambda_L = \Lambda(1,1) \leq 1-\varepsilon < 1$ for some $\varepsilon > 0$. Then according to the homogeneity property of $\Lambda : \Lambda(v,v) \leq (1-\varepsilon)v$ for all

$v \in [0,1]$. Thus, utilizing the monotonicity of Λ in each argument, $\Lambda(u,v) \le \Lambda(v,v) \le (1-\varepsilon/2)u$ for all $u \in [c(\varepsilon)v, v]$ with $c(\varepsilon) = (1-\varepsilon)/(1-\varepsilon/2) < 1$. Hence, using the fact $\Lambda(u,v) \le \min(u,v)$, we have

$$\begin{aligned} \int_{[0,1]^2} \Lambda(u,v)\, duv &= \int_0^1 \left\{ \int_0^{c(\varepsilon v)} \Lambda(u,v)\, du + \int_{c(\varepsilon v)}^{v} \Lambda(u,v)\, du + \int_v^1 \Lambda(u,v)\, du \right\} dv \\ &\le \int_0^1 \left\{ \int_0^{c(\varepsilon v)} u\, du + \int_{c(\varepsilon v)}^{v} \Big(1-\frac{\varepsilon}{2}\Big) u\, du + \int_v^1 v\, du \right\} dv < \frac{1}{3}. \end{aligned}$$

Thus, $\rho_L = 3\int_{[0,1]^2} \Lambda(u,v)\, duv < 1$ which contradicts the assumption $\rho_L = 1$. □

The two best-known families of copulae in theory and practice are the family of Archimedean copulae and the family of elliptical copulae. Thus, we will characterize the tail-dependence measure ρ_L for these two families of copulae via the following two propositions.

Archimedean copulae are described by a continuous, strictly decreasing and convex *generator* function $\phi : [0,\,1] \to [0,\,\infty]$ with $\phi(1) = 0$. The copula C is then given by

$$C(u,v) = \phi^{[-1]}\left(\phi(u) + \phi(v)\right). \tag{13}$$

Here $\phi^{[-1]} : [0,\,\infty] \to [0,\,1]$ denotes the pseudo-inverse of ϕ. The generator ϕ is called strict if $\phi(0) = \infty$ and in this case $\phi^{[-1]} = \phi^{-1}$, see Genest and MacKay (1986), Joe (1997), or Nelsen (1999).

With the exception of $\rho_L = \lambda_L = 0$ and $\rho_L = \lambda_L = 1$, the two measures of tail dependence may well differ. This is, e.g., the case for Archimedean copulae. The following proposition shows that for a large class of Archimedean copulae the tail-dependence coefficients λ_L and ρ_L do not coincide.

Proposition 4 *Let C be a member of the family of bivariate Archimedean copulae with continuously differentiable generator. If the limits ρ_L and $\lambda_L = \lim_{p\downarrow 0} C(p,p)/p = \lim_{p\downarrow 0} 2\phi'(p)/\phi'(C(p,p))$ exist, then*

$$\rho_L = \lim_{p\downarrow 0} \rho(p) = \lambda_L \qquad \Longleftrightarrow \qquad \lambda_L = 0 \quad or \quad \lambda_L = 1.$$

Proof. Consider first the case of an Archimedean copula with non-strict generator ϕ, i.e. $\phi(0) < \infty$. Then according to a remark in Nelsen (1999), p.98, the zero set $\{(u,v) \in [0,1]^2 \mid C(u,v) = 0\}$ has a positive area with boundary curve $\phi(u) + \phi(v) = \phi(0) < \infty$, which is convex. Thus, $C(u,v) = 0$ for all $u, v \in [0,p]$ for p small enough, which implies $\lambda_L = \rho_L = 0$. Further, the equality of λ_L and ρ_L, if either of both is 1 or 0, has been proven in Proposition 3.

Let us now consider an Archimedean copula C with strict generator function ϕ, i.e. $\phi(0) = \infty$. Further, assume $0 < \lambda_L < 1$. The partial derivatives of C exist and are continuous according to the continuous differentiability of the generator function. Then we have

$$\int_0^p \frac{\partial}{\partial p} C(u,p)\, du = \int_0^p \frac{\phi'(p)}{\phi'\{C(u,p)\}}\, du \ge \int_{cp}^p \frac{\phi'(p)}{\phi'\{C(u,p)\}}\, du \ge \frac{\phi'(p)(1-c)p}{\phi'\{C(cp,cp)\}}$$

for any $0 < c < 1$ since $\phi' < 0$ and nondecreasing, and C is nondecreasing in each component. The proof is complete - see Equation (11) - if we find sequences $p_n \downarrow 0$ and $c_n \to c \in (0,1)$ such that $\phi'(p_n)/\phi'(c_n p_n) \ge \varepsilon > 0$. This follows from the fact that $\lim_{p\downarrow 0} C(p,p)/p = \lim_{p\downarrow 0} 2\phi'(p)/\phi'(C(p,p)) = \lambda_L$. Set $c_n := C(p_n,p_n)/p_n \to \lambda_L \in (0,1)$. Then $\phi'(p_n)/\phi'(c_n p_n) \to \lambda_L \in (0,1)$, which proves the proposition. □

Elliptical copulae are the copulae of elliptically contoured distributions such as the multivariate normal distributions, t-distributions, symmetric generalized hyperbolic distributions, or α-stable distributions. In particular, a distribution function F with density function f is elliptically contoured (and non-degenerated) if f possesses the following representation

$$f(\mathbf{x}) = |\Sigma|^{-1/2} h\{(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\}, \qquad \mathbf{x}, \mu \in \mathbb{R}^d,$$

where $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is the *generator*, and $\Sigma \in \mathbb{R}^{d\times d}$ is a positive definite matrix.

Proposition 5 *Let C be the copula of a d-dimensional elliptically contoured distribution with generator h. If the generator is regularly varying (at infinity) with index $-(\alpha+d)/2$, $\alpha > 0$, i.e., $\lim_{t\to\infty} h(ts)/h(t) = s^{-(\alpha+d)/2}$, then*

$$\rho_L = \frac{2\alpha^d(d+1)}{E(B^\alpha)} \int_{[1,\infty]^d} H(\mathbf{x}) \prod_{i=1}^d x_i^{-\alpha-1}\, d\mathbf{x}$$

with function $H(\mathbf{x}) = \int_{\bar{\mathbb{S}}^{d-1}} \min_{i\in I_a} \left\{(\sqrt{\Sigma}\mathbf{a})_i/x_i\right\}^\alpha S(d\mathbf{a})$ with $I_a = \{i \mid (\sqrt{\Sigma}\mathbf{a})_i > 0\}$ and B^2 is $Beta(1/2,(d-1)/2)$ distributed. The space $\bar{\mathbb{S}}^{d-1} := \mathbb{S}^{d-1}\backslash(-\infty,0]^d$, where $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : ||x|| = 1\}$ denotes the $(d-1)$-dimensional unit sphere (regarding the Euclidean norm) and $S(\cdot)$ is the uniform measure on it.

Proof. Without loss of generality we set $\mu = 0$ and the diagonal elements of Σ equal to 1, since elliptical copulae are invariant with respect to these parameters. Then the univariate distributions of the corresponding elliptically-contoured distribution function F coincide and we will denote them with G. We have

$$\rho_L = \lim_{p\downarrow 0} \frac{d+1}{p^{d+1}} \int_{[0,p]^d} C(\mathbf{u})\, d\mathbf{u} = \lim_{p\downarrow 0} \frac{d+1}{p^{d+1}} \int_{(-\infty,G^{-1}(p)]^d} F(\mathbf{x})\, d\left\{\prod_{i=1}^d G(x_i)\right\} = (*).$$

Note that the left endpoint of G equals $-\infty$, because the generator h is regularly varying at infinity. Further, G is continuous and possesses a density if $d \geq 2$ according to Fang, Kotz, and Ng (1990), pp.36. Set $\bar{G} = 1 - G$. Then,

$$(*) = \lim_{t\to\infty} \frac{d+1}{\bar{G}(t)^{d+1}} \int_{(-\infty,-t]^d} F(\mathbf{x})\, d\left\{\prod_{i=1}^d G(x_i)\right\} = \lim_{t\to\infty} (d+1) \int_{[1,\infty)^d} \frac{\bar{F}(t\mathbf{x})}{\bar{G}(t)} \prod_{i=1}^d \frac{tg(tx_i)}{\bar{G}(t)}\, d\mathbf{x}$$

with $\bar{F}(\mathbf{x}) = P(\mathbf{X} > \mathbf{x}) = F(-\mathbf{x})$ and g is the density function of G which is symmetric. Utilizing Propositions 3.4 and 3.7 in Schmidt (2002), the tail function $\bar{G}$ is regularly varying at infinity with index $-\alpha$, $\alpha > 0$. Further, the corresponding density function g is regularly varying at infinity with index $-\alpha - 1$ because g is monotone on $(0,\infty)$ by a result in Fang, Kotz, and Ng (1990), p.37, and Theorem 1.7.2 in Bingham, Goldie, and Teugels (1987). Moreover, by Theorem 1.5.2 in the latter reference, the convergence $g(tx)/g(t) \to x^{-\alpha-1}$ is uniform in $x \in [1,\infty)$. This implies that $tg(tx)/\bar{G}(t) \to \alpha \cdot x^{-\alpha-1}$ uniformly in $x \in [1,\infty)$.

It remains to be shown that

$$\bar{F}(t\mathbf{x})/\bar{G}(t) \to 2 \cdot H(\mathbf{x})/E(B^\alpha)$$

uniformly in $\mathbf{x} \in [1,\infty)^d$ with function $H(\mathbf{x}) = \int_{\bar{\mathbb{S}}^{d-1}} \min_{i\in I_a} \left\{(\sqrt{\Sigma}\mathbf{a})_i/x_i\right\}^\alpha S(d\mathbf{a})$ and $I_a = \{i \mid (\sqrt{\Sigma}\mathbf{a})_i > 0\}$ The random variable B^2 is $Beta(1/2,(d-1)/2)$ distributed. We utilize the following stochastic representation of elliptically contoured distributions with location $\mu = \mathbf{0}$:

$$\mathbf{X} \stackrel{d}{=} R_d\sqrt{\Sigma}\mathbf{U}^{(d)}$$

with $\sqrt{\Sigma}\sqrt{\Sigma}' = \Sigma$. The random variable $R_d \geq 0$ is stochastically independent of the d-dimensional random vector $\mathbf{U}^{(d)}$ which is uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$. Further, G is the distribution function of a random variable $R_1 U$, where $R_1 \geq 0$ is stochastically independent of the Bernoulli random variable U. Additionally, Proposition 3.1 in Schmidt (2002) and the related proof imply that $P(R_1 > t)/P(R_d > t) \to E(B^\alpha)$ as $t \to \infty$. For $t \geq 0$ we have

$$\frac{\bar{F}(t\mathbf{x})}{\bar{G}(t)} = \frac{P(R_d\sqrt{\Sigma}\mathbf{U}^{(d)} > t\mathbf{x})}{P(R_1 U > t)} = 2 \cdot \frac{P(R_d > t)}{P(R_1 > t)} \int_{\bar{\mathbb{S}}^{d-1}} \frac{P(R_d\sqrt{\Sigma}\mathbf{a} > t\mathbf{x})}{P(R_d > t)}\, S(d\mathbf{a}). \qquad (14)$$

Due to the uniform convergence of $P(R_d > tx)/P(R_d > t) \to x^{-\alpha}$ in $x \in [\varepsilon, \infty)$ for any fixed $\varepsilon > 0$ by Proposition 3.1 of the last reference and Theorem 1.5.2 in Bingham, Goldie, and Teugels (1987), we obtain

$$\frac{P(R_d\sqrt{\Sigma}\mathbf{a} > t\mathbf{x})}{P(R_d > t)} = \frac{P\big[R_d > t \cdot \max_{i \in I_a} \{x_i/(\sqrt{\Sigma}\mathbf{a})_i\}\big]}{P(R_d > t)} \to \min_{i \in I_a} \big\{(\sqrt{\Sigma}\mathbf{a})_i/x_i\big\}^{\alpha},$$

which converges uniformly in $\mathbf{a} \in \overline{\mathbb{S}}^{d-1}$ and $\mathbf{x} \in [1,\infty)^d$. Combining this with Formula (14) yields the desired result. □

Examples. i) Consider the copulae of d-dimensional symmetric Pearson-type VII distributions. These distributions are elliptically contoured and its generator has the form

$$h(t) = c_d\Big(1 + \frac{t}{\alpha}\Big)^{-N}, \quad N > d/2,\ \alpha > 0, \tag{15}$$

where c_d denotes a normalizing constant. Obviously the generator given in (15) is regularly varying with index $-N$, and Proposition 5 is applicable. Setting $N = (d+\alpha)/2$ in (15) yields the copulae of d-dimensional t-distributions, which include the copulae of multivariate Cauchy distributions for $\alpha = 1$.

ii) The copulae of d-dimensional normal distributions do not possess a regularly varying generator. It is well-known that for dimension $d = 2$ these copulae are tail independent, i.e. $\lambda_L = 0$. Thus, the tail copula $\Lambda_L \equiv 0$ by Theorem 1.iv in Schmidt and Stadtmüller (2006). Hence, the formulae after Equation (8) imply that $\rho_L = 0$, see also Proposition 3. For arbitrary dimension d, one can similarly show that $\rho_L = 0$.

Another choice of the weighting function g, which takes the form $g(\mathbf{u}) = \mathbf{1}_{[1-p,1]^d}(\mathbf{u}),\ 0 < p \leq 1$, provides us with a multivariate measure of upper tail dependence, defined by $\rho_U := \lim_{p\downarrow 0} \rho^\star(p)$ with $\rho^\star(p) := \rho\{\mathbf{1}_{[1-p,1]^d}(\mathbf{u})\}$. For dimension $d = 2$, this measure ρ_U is an alternative of the so-called upper tail-dependence coefficient which is defined by $\lambda_U := \lim_{p\downarrow 0} P\{X_1 > F_{X_1}^{-1}(1-p) \mid X_2 > F_{X_2}^{-1}(1-p)\}$, if existent, and thus represents an analogue to λ_L. The properties and relationship between λ_U and ρ_U can be derived in a similar manner as the previous results. We mention that for radially symmetric bivariate copulae such as bivariate elliptical copulae, i.e. where $C(u,v) = u + v - 1 + C(1-u, 1-v)$, the measures ρ_U and ρ_L coincide.

4 Statistical Inference

Statistical inference for the dependence measures introduced in Section 3 can be developed under the assumption of known or unknown marginal distributions. The case of known marginal distributions is given for the sake of completeness since - as we have already mentioned - this case is usually not of practical relevance.

4.1 Estimation under known marginal distributions

Consider a random sample $(\mathbf{X}_j)_{j=1,\ldots,n}$ from a d-dimensional random vector $\mathbf{X}$ with joint distribution function F and copula C. In the present section, we assume that the univariate marginal distribution functions F_i of F are continuous and known. We set $U_{ij} = F_i(X_{ij}),\ i = 1,\ldots,d,\ j = 1,\ldots,n$. Thus, the random vectors $\mathbf{U}_j = (U_{1j},\ldots,U_{dj}),\ j = 1,\ldots,n$, are distributed according to the copula C. The assumption of known marginal distributions is dropped in the next section. The reason for making this assumption here lies in the following theorem. It shows that under fairly general assumptions on the weighting function g the natural estimator of $\rho(g)$ converges weakly to a Gaussian field. Moreover, the convergence is uniform in g. The estimator we utilize is of the form

$$\hat{\rho}^\star(g) := \Bigg\{\frac{1}{n}\sum_{j=1}^{n}\int_{\mathbf{U_j} \leq \mathbf{u}} g(\mathbf{u})\, d\mathbf{u} - \bar{c}(g)\Bigg\}\Big/c(g), \tag{16}$$

where $\bar{c}(g) := \int_{[0,1]^d} \Pi(\mathbf{u})g(\mathbf{u})\ d\mathbf{u}$ and $c(g) := \int_{[0,1]^d} M(\mathbf{u})g(\mathbf{u})\ d\mathbf{u} - \bar{c}(g)$. From (16), we obtain below an estimator for $\rho(p)$ as a special case:

$$\hat{\rho}^\star(p) := \left\{ \frac{1}{n} \sum_{j=1}^{n} \prod_{i=1}^{d} \left(p - U_{ij}\right)^+ - \left(\frac{p^2}{2}\right)^d \right\} \Big/ \left\{ \frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2}\right)^d \right\}. \tag{17}$$

Theorem 6 (Asymptotic normality under known marginal distributions) *Suppose F is a d-dimensional distribution function with continuous marginal distribution functions and copula C. Let $\hat{\rho}^\star(g)$ be the estimator defined in (16), where $g \in \mathcal{F}_K$ and $\mathcal{F}_K$ is the set of all integrable functions $h : [0,1]^d \mapsto \mathbb{R}_+$ with $\int_{[0,1]^d} h(\mathbf{u})/c(h)\ d\mathbf{u}$ being bounded by $K > 0$. Then, the empirical process converges weakly,*

$$\sqrt{n}(\hat{\rho}^\star(g) - \rho(g)) \xrightarrow{w} \mathbb{G}^\star(g),$$

where $\mathbb{G}^\star(g)$ is a centered tight continuous Gaussian random field. Weak convergence takes place in $\ell^\infty(\mathcal{F}_K)$. The covariance structure of $\mathbb{G}^\star$ is given by

$$I\!\!E\big\{\mathbb{G}^\star(g) \cdot \mathbb{G}^\star(g')\big\} = \frac{1}{c(g) \cdot c(g')} \int_{[0,1]^{2d}} \Big[C\{\min(\mathbf{u},\mathbf{v})\} - C(\mathbf{u})C(\mathbf{v})\Big] g(\mathbf{u})g'(\mathbf{v})\ d\mathbf{u}\mathbf{v}.$$

Proof. Consider the collection $\mathcal{F}$ of measurable functions $f : \mathbb{R}^d \mapsto \mathbb{R}_+$ given by

$$f(\mathbf{x}) = \int_{\mathbf{x} \le \mathbf{u}} g(\mathbf{u})/c(g)\ d\mathbf{u} \quad \text{with } g \in \mathcal{F}_K.$$

Using the notation $\mu f = \int f\ d\mu$ for a signed measure μ and an integrable function f, we rewrite the $\mathcal{F}$-indexed empirical process of the estimator (16) as $\sqrt{n}(\mathbb{P}_n - P)f = \sum_{j=1}^n \{f(\mathbf{U_j}) - Pf\}/\sqrt{n}$, where P is the distribution of $\mathbf{U}$ and $\mathbb{P}_n$ denotes the empirical measure. Note that the prerequisites imply that $\sup_{f\in\mathcal{F}} |f(\mathbf{x}) - pf| < \infty$ for every x. Thus, the empirical process $\{\sqrt{n}(\mathbb{P}_n - P)f : f \in \mathcal{F}\}$ can be viewed as a map into $\ell^\infty(\mathcal{F})$. We may identify $\ell^\infty(\mathcal{F})$ with the space $\ell^\infty(\mathcal{F}_K)$. Thus, we have to show weak convergence of $\sqrt{n}(\mathbb{P}_n - P)f$ to the limit $\mathbb{G}^\star$ in $\ell^\infty(\mathcal{F}_K)$, where $\mathbb{G}^\star$ is a tight Borel measurable element in $\ell^\infty(\mathcal{F}_K)$. In this case, $\mathcal{F}_K$ is called a Donsker class.

It is well known that the set of indicator functions on cells $[\mathbf{x}, \infty)$ is a Vapnik-Cervonenkis class of functions and, thus, a Donsker class (see, for instance, Example 2.6.1 in Van der Vaart and Wellner (1996)). Further, the convex hull of this set of functions coincides with the set of discrete distribution functions with finitely many atoms and is, thus, also a Donsker class according to Theorem 2.10.3 in Van der Vaart and Wellner (1996). Utilizing the Glivenko-Cantelli theorem, we conclude that the set of distribution functions is a Donsker class since the modulus of continuity does not increase for this class. By rescaling, we may assume without loss of generality that $\mathbf{x} \mapsto \int_{\mathbf{x}\le\mathbf{u}} g(\mathbf{u})/c(g)\ d\mathbf{u}$ is uniformly bounded by 1. Thus, $\mathcal{F}_K$ is a subset of the convex hull of all distribution functions and, hence, it is a Donsker class by the last mentioned theorem.

As usual, the assertion that the process $\mathbb{G}^\star$ is Gaussian follows from the multivariate central limit theorem as does the fact that the functions $f \in \mathcal{F}$ are square integrable. □

Theorem 6 immediately yields a useful result for $\hat{\rho}^\star(p)$.

Corollary 7 *Let $\hat{\rho}^\star(p)$ be the estimator, as defined in (17). Then*

$$\sqrt{n}(\hat{\rho}^\star(p) - \rho(p)) \xrightarrow{w} \mathbb{G}^\star(p),$$

where $\mathbb{G}^\star(p)$ is a centered tight continuous Gaussian process. Weak convergence takes place in $\ell^\infty((\varepsilon, 1])$ for any fixed $\varepsilon > 0$. The covariance structure of $\mathbb{G}^\star$ is given by

$$I\!\!E\big\{\mathbb{G}^\star(p) \cdot \mathbb{G}^\star(q)\big\} = \frac{1}{c(p) \cdot c(q)} \int_{[0,p]\times[0,q]} C\{\min(\mathbf{u},\mathbf{v})\} - C(\mathbf{u})C(\mathbf{v})\ d\mathbf{u}\mathbf{v}$$

with $c(p) := \frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2}\right)^d$ and $c(q) = \frac{q^{d+1}}{d+1} - \left(\frac{q^2}{2}\right)^d$.

Remark. Weak convergence in Theorem 6 can be extended to the set of all functions in $\tilde{\mathcal{F}} = \bigcup_{K=1}^{\infty} \mathcal{F}_K$ by utilizing the metric $d(f, f') = \sum_{K=1}^{\infty} \big(||f - f'||_{\mathcal{F}_K} \wedge 1\big) 2^{-K}$. In this context, weak convergence in Corollary 7 takes place in $\ell^{\infty}(\bigcup_{n=1}^{\infty}(1/n, 1])$. However, weak convergence of $\sqrt{n}(\hat{\rho}^{\star}(p) - \rho(p))$ in $\ell^{\infty}([0,1])$ is not straightforward. This problem is addressed in Section 4.3 in more detail.

4.2 Estimation under unknown marginal distributions

We now assume that the univariate marginal distribution functions F_{X_i} of F are continuous but unknown. The marginal distribution functions F_{X_i} are estimated by their empirical counterparts

$$\hat{F}_{i,n}(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{X_{ij} \leq x\}}, \quad \text{for } i = 1, \ldots, d \text{ and } x \in \mathbb{R}.$$

Further, set $\hat{U}_{ij,n} := \hat{F}_{i,n}(X_{ij})$ for $i = 1, \ldots, d,\ j = 1, \ldots, n$, and $\hat{\mathbf{U}}_{j,n} = (\hat{U}_{1j,n}, \ldots, \hat{U}_{dj,n})$. Note that

$$\hat{U}_{ij,n} = \frac{1}{n}(\text{rank of } \ X_{ij} \ \text{ in } \ X_{i1}, \ldots, X_{in}).$$

The estimation of $\rho(g)$ will therefore be based on ranks (and not on the observations itself). In other words, we consider rank order statistics. The copula C is estimated by the empirical copula which is defined as

$$\hat{C}_n(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d} \mathbf{1}_{\{\hat{U}_{ij,n} \leq u_i\}} \quad \text{for} \quad \mathbf{u} = (u_1, \ldots, u_d)' \in [0,1]^d.$$

We define the following nonparametric estimator for $\rho(g)$:

$$\hat{\rho}(g) := \left\{ \frac{1}{n}\sum_{j=1}^{n} \int_{\hat{\mathbf{U}}_{\mathbf{j},\mathbf{n}} \leq \mathbf{u}} g(\mathbf{u})\ d\mathbf{u} - \bar{c}(g) \right\} / c(g), \tag{18}$$

where $\bar{c}(g) := \int_{[0,1]^d} \Pi(\mathbf{u}) g(\mathbf{u})\ d\mathbf{u}$ and $c(g) := \int_{[0,1]^d} M(\mathbf{u}) g(\mathbf{u})\ d\mathbf{u} - \bar{c}(g)$. In order to derive the asymptotic behavior of the above estimator, we use the following theorem. For a *proof* and further discussion see Rüschendorf (1976), Stute (1984), Gänßler and Stute (1987), Fermanian, Radulović, and Wegkamp (2004), and Tsukahara (2005).

Theorem 8 *Let F be a continuous d-dimensional distribution function with copula C. Under the additional assumption that the partial derivatives $D_iC(\mathbf{u})$ exist and are continuous for $i = 1, \ldots, d$, we have*

$$\sqrt{n}(\hat{C}_n(\mathbf{u}) - C(\mathbf{u})) \xrightarrow{w} \mathbb{G}_C(\mathbf{u}).$$

Weak convergence takes place in $\ell^{\infty}([0,1]^d)$ and $\mathbb{G}_C(\mathbf{u}) = \mathbb{B}_C(\mathbf{u}) - \sum_{i=1}^{d} D_iC(\mathbf{u})\mathbb{B}_C(\mathbf{u}^{(i)})$ with D_i denoting the i-th partial derivative. The process $\mathbb{B}_C$ is a tight centered Gaussian process on $[0,1]^d$ with covariance function $E\{\mathbb{B}_C(\mathbf{u})\mathbb{B}_C(\mathbf{v})\} = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v})$, i.e., $\mathbb{B}_C$ is a d-dimensional Brownian sheet. The vector $\mathbf{u}^{(i)}$ denotes the vector where all coordinates, except the i-th coordinate of $\mathbf{u}$, are replaced by 1.

We can now prove asymptotic normality of the above estimator.

Theorem 9 (Asymptotic normality) *Let $\hat{\rho}(g)$ be the estimator defined in* (18)*, where $g \in \mathcal{F}_M$ and $\mathcal{F}_M$ is the set of all integrable functions $h : [0,1]^d \mapsto \mathbb{R}_+$ with $h(\mathbf{u})/c(h)$ being uniformly bounded by $M > 0$. Then, under the assumptions of Theorem 8, the process*

$$\sqrt{n}(\hat{\rho}(g) - \rho(g)) \xrightarrow{w} \mathbb{G}(g),$$

where $\mathbb{G}(g)$ is a centered tight continuous Gaussian random field. Weak convergence takes place in $\ell^\infty(\mathcal{F}_M)$. With $\mathbb{G}_C$ as in Theorem 8 we have

$$\mathbb{G}(g) = \frac{1}{c(g)} \int_{[0,1]^d} \mathbb{G}_C(\mathbf{u}) g(\mathbf{u})\, d\mathbf{u}.$$

Proof. Weak convergence in $\ell^\infty(\mathcal{F}_M)$ to a tight limit is equivalent to marginal convergence plus asymptotic tightness, see Theorem 1.5.4 in Van der Vaart and Wellner (1996). The estimator $\hat{\rho}(g)$ and $\rho(g)$ are obviously invariant with respect to any scaling of the weighting function g by some constant $c \neq 0$. Thus, we restrict ourselves to the space $\mathcal{F}_M$ which consists of scaling-invariant weighting functions, e.g., where $\int g(\mathbf{u})\, d\mathbf{u} = 1$. We equip this space $\mathcal{F}_M$ with the metric $d(g, g') = ||g/c(g) - g'/c(g')||_{[0,1]^d}$ with $||h||_{[0,1]^d} := \sup_{[0,1]^d} |h(\mathbf{u})|$. Define $X_n(g) := \sqrt{n}(\hat{\rho}(g) - \rho(g))$. Marginal convergence is given if $(X_n(g_1), \ldots, X_n(g_k))$ converges weakly for every finite subset of functions $g_1, \ldots, g_k$ in $\mathcal{F}_M$. This follows from Theorem 8 and the continuous mapping theorem, because the integral operator is a continuous linear map on $\ell^\infty([0,1]^d)$ into $\mathbb{R}$ and $\mathbb{G}_C$ is a tight Gaussian process. The resulting limiting vector is normally distributed. This proposition is verified by writing the integration as the limit of projection maps Φ_m which are multivariate normal, since $\mathbb{G}_C$ is a Gaussian process. An application of the uniqueness theorem of characteristic functions proves the assertion.

Asymptotic tightness still remains to be shown. We choose a version of the limiting Gaussian process $\mathbb{G}(g)$ which is a tight map in $\mathcal{F}_M$. Then, it remains to be shown that $X_n(g)$ is asymptotically uniformly equicontinuous in probability. For every $\varepsilon, \eta > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned}
&P\Big(\sup_{d(g,g')<\delta} |X_n(g) - X_n(g')| > \varepsilon\Big) \\
&\quad \leq P\left(\sup_{d(g,g')<\delta} \int_{[0,1]^d} \left|\sqrt{n}\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\}\right| \cdot \left|\frac{g(\mathbf{u})}{c(g)} - \frac{g'(\mathbf{u})}{c(g')}\right| d\mathbf{u} > \varepsilon\right) \\
&\quad \leq P\left(\int_{[0,1]^d} \left|\sqrt{n}\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\}\right| d\mathbf{u} > \frac{\varepsilon}{\delta}\right) < \eta,
\end{aligned}$$

due to the weak convergence of $\sqrt{n}\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})$ towards a tight centered Gaussian process in $\ell^\infty([0,1]^d)$ (Theorem 8) and an application of the continuous mapping theorem. □

Corollary 10 *Consider the following estimator for $\rho(p)$:*

$$\hat{\rho}_n(p) = \left\{\frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}(p - \hat{U}_{ij,n})^+ - \Big(\frac{p^2}{2}\Big)^d\right\} \Big/ \Big\{\frac{p^{d+1}}{d+1} - \Big(\frac{p^2}{2}\Big)^d\Big\}. \tag{19}$$

Under the assumptions of Theorem 9 we have

$$\sqrt{n}\{\hat{\rho}_n(p) - \rho(p)\} \xrightarrow{w} \mathbb{G}(p) = \int\limits_{[0,p]^d} \mathbb{G}_C(\mathbf{u})\, d\mathbf{u} \Big/ \Big\{\frac{p^{d+1}}{d+1} - \Big(\frac{p^2}{2}\Big)^d\Big\},$$

where $\mathbb{G}(p)$ is a centered Gaussian process and $\mathbb{G}_C$ is given in Theorem 8. Weak convergence takes place in $\ell^\infty([\varepsilon, 1])$ for arbitrary but fixed $0 < \varepsilon < 1$. The covariance structure of $\mathbb{G}(p)$ is given by

$$E\{\mathbb{G}(p)\mathbb{G}(q)\} = \frac{1}{c(p)\cdot c(q)} \int\limits_{[0,p]^d\times[0,q]^d} E\{\mathbb{G}_C(\mathbf{u}) \cdot \mathbb{G}_C(\mathbf{v})\}\, d\mathbf{u}\mathbf{v}, \quad 0 < p, q \leq 1, \tag{20}$$

with $c(p) = \frac{p^{d+1}}{d+1} - \big(\frac{p^2}{2}\big)^d$.

It is interesting to look at the special case $p = 1$ and $d = 2$. Here, we obtain

$$\hat{\rho}_n(1) = 12\Big(\frac{1}{n}\sum_{j=1}^{n}\hat{U}_{1j,n}\hat{U}_{2j,n} - \frac{1}{n}\Big) - 3$$

which is an alternative estimator for Spearman's ρ_S for $d = 2$. Note that $\hat{\rho}_n(1)$ is slightly different from the traditional rank order statistics of Spearman's rho

$$\hat{\rho}'_n = 1 - \frac{6n}{n^2 - 1}\sum_{j=1}^{n}(\hat{U}_{1j,n} - \hat{U}_{2j,n})^2,$$

which is used if no ties are present in the sample. It can be shown that $\hat{\rho}_n(1) \leq \hat{\rho}'_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty}\sqrt{n}\{\hat{\rho}_n(1) - \hat{\rho}'_n\} = 0$ with probability one. Therefore $\hat{\rho}_n(1)$ and $\hat{\rho}'_n$ have the same asymptotic distribution. The asymptotic variance of $\sqrt{n}\{\hat{\rho}_n(1) - \rho_S\}$ is given by

$$asyVar\big\{\hat{\rho}_n(1)\big\} = 144\int\limits_{[0,1]^2}\int\limits_{[0,1]^2} E\Big\{\mathbb{G}_C(\mathbf{u})\mathbb{G}_C(\mathbf{v})\Big\}\, d\mathbf{u}d\mathbf{v}, \tag{21}$$

which is also established in, e.g., Rüschendorf (1976) and Genest and Rémillard (2004). Explicit formulas for the asymptotic variance (21) for different families of copulae are calculated in Schmid and Schmidt (2006). It turns out that for most copulae it is difficult to estimate the asymptotic covariance structure (20) or the asymptotic variance (21). Fortunately, the following bootstrap result holds. Here, $(\mathbf{X}_j^B)_{j=1,\ldots,n}$ denotes the bootstrap sample which is obtained by sampling from $(\mathbf{X}_j)_{j=1,\ldots,n}$ with replacement. The empirical copula of $(\mathbf{X}_j^B)_{j=1,\ldots,n}$ is denoted by $\hat{C}_n^B$.

Theorem 11 (The bootstrap) *Let $\mathcal{F}_M$ be the set of all integrable functions $h : [0,1]^d \mapsto \mathbb{R}_+$ with $h(\mathbf{u})/c(h)$ being uniformly bounded by $M > 0$. Suppose $\hat{\rho}_n(g)$ is the estimator defined in (18) and $\hat{\rho}_n^B(g)$ denotes the corresponding estimator for the bootstrap sample $(\mathbf{X}_j^B)_{j=1,\ldots,n}$. Then, under the assumptions of Theorem 8, the process $\sqrt{n}\{\hat{\rho}_n^B(g) - \hat{\rho}_n(g)\}$, $g \in \mathcal{F}_M$, converges weakly to the same Gaussian process as $\sqrt{n}\{\hat{\rho}_n(g) - \rho(g)\}$, $g \in \mathcal{F}_M$, with probability one. Weak convergence takes place in $\ell^\infty(\mathcal{F}_M)$.*

Proof. Set $X_n(g) := \sqrt{n}(\hat{\rho}_n(g) - \rho(g))$ and $Y_n(g) := \sqrt{n}(\hat{\rho}_n^B(g) - \hat{\rho}_n(g))$. The multidimensional extension of Theorem 5 in Fermanian, Radulović, and Wegkamp (2004) implies that the process $\sqrt{n}\{\hat{C}_n^B - \hat{C}_n\}$ converges weakly to the same Gaussian process as $\sqrt{n}\{\hat{C}_n - C\}$ with probability 1. Weak convergence takes place in $\ell^\infty([0,1]^2)$. Thus, for every finite subset of functions $g_1,\ldots,g_k$ in $\mathcal{F}_M$, $(Y_n(g_1),\ldots,Y_n(g_k))$ converges weakly to the same limit as $(X_n(g_1),\ldots,X_n(g_k))$ by the continuous mapping theorem. Asymptotic tightness of $Y_n(\cdot)$ follows by the same arguments as given in the proof of Theorem 9. □

4.3 Nonparametric Estimation of ρ_L

The present section discusses the asymptotic behavior of $\hat{\rho}_n(p)$, as defined in (19), if p tends to zero. In particular, we consider the following estimator for the multivariate measure of (lower) tail dependence ρ_L defined in (7):

$$\hat{\rho}_{L,n} := \hat{\rho}_n(k/n)$$

with some parameter $k \in \{1,\ldots,n\}$ to be chosen by the statistician. For the asymptotic results we assume throughout this section that $k = k(n) \to \infty$ and $k/n \to 0$ as $n \to \infty$.

Condition 12 (Second order condition) *Let C be a copula. Assume the lower tail copula $\Lambda_L(\mathbf{u}) := \lim_{p\downarrow 0} C(p\cdot\mathbf{u})/p \not\equiv 0$ exists everywhere on $[0,1]^d$. Then $\Lambda_L(\mathbf{u})$ is said to satisfy a second order condition if a function $A : \mathbb{R}_+ \to \mathbb{R}_+$ exists such that $A(1/p) \to 0$ as $p \downarrow 0$ and*

$$\lim_{p\downarrow 0}\frac{\Lambda_L(\mathbf{u}) - C(p\cdot\mathbf{u})/p}{A(1/p)} = g(\mathbf{u}) < \infty, \quad \mathbf{u} \in [0,1]^d,$$

where the convergence is locally uniformly and the function g is nonconstant.

Note that $A(\cdot)$ is regularly varying at infinity so this is just a second order condition on regular variation, see de Haan and Stadtmüller (1996).

Theorem 13 (Asymptotic normality of $\hat{\rho}_{L,n}$) *Let F be a d-dimensional distribution function with continuous marginal distribution functions. If the Second order Condition 12 holds and the therein mentioned tail copula Λ_L possesses continuous partial derivatives, then for*

$$\sqrt{k}A(n/k) \to 0 \quad \text{and} \quad \sqrt{k}(k/n)^{d-1} \to 0 \quad \text{as} \quad n \to \infty \tag{22}$$

we have

$$\sqrt{k}\big(\hat{\rho}_{L,n} - \rho_L\big) \xrightarrow{d} N(0, \sigma_{\Lambda_L}) \quad \text{as } n \to \infty,$$

with asymptotic variance $\sigma_{\Lambda_L} = (d+1)\int_{[0,1]^d} \mathbb{G}_{\Lambda_L}(\mathbf{u})\, d\mathbf{u}$. The process $\mathbb{G}_{\Lambda_L}$ is a centered tight continuous Gaussian random field (a detailed specification is given in the proof).

Proof. Dominated convergence yields

$$\rho_L = \lim_{p \downarrow 0} (d+1) \int_{[0,1]^d} \frac{C(p \cdot \mathbf{u})}{p}\, d\mathbf{u} = (d+1)\int_{[0,1]^d} \Lambda_L(\mathbf{u})\, d\mathbf{u},$$

where Λ_L is the (lower) tail copula. Thus, we may write

$$\sqrt{k}\big(\hat{\rho}_{L,n} - \rho_L\big) = (d+1)\int_{[0,1]^d} \sqrt{k}\Big\{\frac{n}{k}\hat{C}_n\Big(\frac{k}{n}\cdot \mathbf{u}\Big) - \Lambda_L(\mathbf{u})\Big\}\, d\mathbf{u} + \sqrt{k}O(1)\Big(\frac{k}{n}\Big)^{d-1},$$

with empirical copula $\hat{C}_n$. Because of (22), the last term in the above equation vanishes and it suffices to prove that

$$\sqrt{k}\Big\{\frac{n}{k}\hat{C}_n\Big(\frac{k}{n}\cdot \mathbf{u}\Big) - \Lambda_L(\mathbf{u})\Big\} \xrightarrow{w} \mathbb{G}_{\Lambda_L}(\mathbf{u}) \quad \text{as } n \to \infty$$

with $k = k(n) \to \infty$ and $k/n \to 0$ as $n \to \infty$, and some centered tight Gaussian process $\mathbb{G}_{\Lambda_L}$. Precisely this has been shown in Schmidt and Stadtmüller (2006), Theorem 5, where $n/k\cdot\hat{C}_n(k/n\cdot\mathbf{u})$ is called the *empirical tail copula* which they denote $\hat{\Lambda}_{L,n}(\mathbf{u})$. The derivations in this reference are based on a slight modification of the notion of an empirical copula. The asymptotic results, however, are equivalent, see Section 4 in this reference. The limiting process $\mathbb{G}_{\Lambda_L}(\mathbf{u})$ can be expressed by

$$\mathbb{G}_{\Lambda_L}(\mathbf{u}) = \mathbb{G}^*_{\Lambda_L}(\mathbf{u}) - \sum_{i=1}^{d} D_i\Lambda_L(\mathbf{u})\mathbb{G}^*_{\Lambda_L}(\mathbf{u}^{\{i\}})$$

with D_i denoting the i-th partial derivative and $\mathbb{G}^*_{\Lambda_L}$ being a centered tight continuous Gaussian process. The vector $\mathbf{u}^{\{i\}}$ corresponds to a vector where all coordinates, except the i-th coordinate of $\mathbf{u}$, are replaced by ∞. Weak convergence takes place in $\ell^\infty([0,1]^d)$. The covariance structure of $\mathbb{G}^*_{\Lambda_L}$ is given by

$$I\!\!E\big\{\mathbb{G}^*_{\Lambda_L}(\mathbf{u}) \cdot \mathbb{G}^*_{\Lambda_L}(\mathbf{v})\big\} = \Lambda_L(\mathbf{u} \wedge \mathbf{v})$$

for $\mathbf{u}, \mathbf{v} \in [0,1]^d$. The proof is finished by an application of the continuous mapping theorem. □

Acknowledgment

The authors are grateful to the editor and the referees for comments which led to an improvement of the paper.

References

Abdous, B., A.L. Fougères, and K. Ghoudi, 2005, Extreme behaviour for bivariate elliptical distributions, *Canadian Journal of Statistics* 33(3), 317–334.

Bingham, N.H., C.M. Goldie, and J.L. Teugels, 1987, *Regular variation* vol. 27 of *Encyclopedia of Mathematics and its Applications.* (Cambridge University Press, Cambridge).

Campbell, R., K. Koedijk, and P. Kofman, 2002, Increased Correlation in Bear Markets, *Financial Analysts Journal* Jan-Feb, 87–94.

de Haan, L., and U. Stadtmüller, 1996, Generalized regular variation of second order, *J. Australian Mathematical Society* 61, 381–395.

Embrechts, P., A. McNeil, and D. Straumann, 2002, Correlation and Dependency in Risk Management: Properties and Pitfalls, *In: Risk Management: Value at Risk and Beyond, ed. M.A.H. Dempster, Cambridge University Press, Cambridge,* p. 176–223.

Fang, K.T., S. Kotz, and K.W. Ng, 1990, *Symmetric Multivariate and Related Distributions.* (Chapman and Hall, London).

Fermanian, J.D., D. Radulović, and M. Wegkamp, 2004, Weak convergence of empirical copula processes, *Bernoulli* 10(5), 847–860.

Forbes, J.F., and R. Rigobon, 2002, No contagion, only interdependence: Measuring stock market comovements, *The Journal of Finance* LVII(5), 2223–2261.

Frahm, G., M. Junker, and R. Schmidt, 2005, Estimating the tail-dependence coefficient: Properties and Pitfalls, *Insurance: Mathematics and Economics* 37, 80–100.

Gänßler, P., and W. Stute, 1987, *Seminar on Empirical Processes, DMV-Seminar Vol. 9.* (Birkhäuser, Basel).

Genest, C., and R.J. MacKay, 1986, Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données, *Canadian Journal of Statistics* 14, 145–159.

Genest, C., and B. Rémillard, 2004, Tests of Independence and Randomness Based on the Empirical Copula Process, *Test* 13(2), 335–369.

Joe, H., 1990, Multivariate Concordance, *Journal of Multivariate Analysis* 35, 12–30.

Joe, H., 1997, *Multivariate Models and Dependence Concepts.* (Chapman and Hall London).

Nelsen, R.B., 1996, Nonparametric measures of multivariate association, *In: Distribution with Fixed Marginals and Related Topics, IMS Lecture Notes - Monograph Series* 28, 223–232.

Nelsen, R.B., 1999, *An Introduction to Copulas.* (Springer Verlag, New York).

Rüschendorf, L., 1976, Asymptotic normality of multivariate rank order statistics, *Annals of Statistics* 4, 912–923.

Ruymgaart, F.H., and M.C.A. van Zuijlen, 1978, Asymptotic normality of multivariate linear rank statistics in the non-i.i.d. case, *Annals of Statistics* 6(3), 588–602.

Schlather, M., 2001, Examples for the coefficient of tail dependence and the domain of attraction of a bivariate extreme value distribution, *Statist. Probab. Lett.* 53(3), 325–329.

Schmid, F., and R. Schmidt, 2006, On the asymptotic behavior of Spearman's rho and related multivariate extensions, *Statist. Probab. Lett.*, to appear.

Schmidt, R., 2002, Tail dependence for elliptically contoured distributions, *Math. Methods of Operations Research* 55(2), 301–327.

Schmidt, R., and U. Stadtmüller, 2006, Nonparametric estimation of tail dependence, *Scandinavian Journal of Statistics* 33, 307-335.

Schweizer, B., and E.F. Wolff, 1981, On nonparametric measures of dependence for random variables, *Annals of Statistics* 9, 870–885.

Sibuya, M., 1960, Bivariate extreme statistics, *Ann. Inst. Statist. Math.* 11, 195–210.

Sklar, A., 1959, Fonctions de répartition à n dimensions et leurs marges, *Publ. Inst. Statist. Univ. Paris* 8, 229–231.

Spearman, C., 1904, The proof and measurement of association between two things, *Amer. J. Psych.* 15, 72–101.

Stute, W., 1984, The oscillation behavior of empirical processes: The multivariate case, *Annals of Probability* 12, 361–379.

Tsukahara, H., 2005, Semiparametric estimation in copula models, *Canadian Journal of Statistics* 33(3), 357–375.

Van der Vaart, A.W., and J.A. Wellner, 1996, *Weak Convergence and Empirical Processes.* (Springer Verlag, New York).

Wolff, E.F., 1980, N-dimensional measures of dependence, *Stochastica* 4(3), 175–188.