CREDIT RISK MODELLING AND ESTIMATION VIA ELLIPTICAL COPULAE

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Abstract. Dependence modelling plays a crucial role within internal credit risk models. The theory of copulae, which describes the dependence structure between a multi-dimensional distribution function and the corresponding marginal distributions, provides useful tools for dependence modelling. The difficulty in employing copulae for internal credit risk models arises from the appropriate choice of a copula function. From the practical point of view the dependence modelling of extremal credit default events turns out to be a desired copula property. This property can be modelled by the so-called tail dependence concept, which describes the amount of dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a bivariate distribution. We will give a characterization of tail dependence via a tail dependence coefficient for the class of elliptical copulae. This copula class inherits the multivariate normal, t, logistic, and symmetric general hyperbolic copula. Further we embed the concepts of tail dependence and elliptical copulae into the framework of extreme value theory. Finally we provide a parametric and non-parametric estimator for the tail dependence coefficient.

Published in: G. Bohl, G. Nakhaeizadeh, S.T. Rachev, T. Ridder and K.H. Vollmer (eds.), *Credit Risk - Measurement, Evaluation and Management*, Physica-Verlag Heidelberg, 2003, 267-289.

Contents

Introduction

The New Basel Capital Accord [16] will be the next stepping-stone for regulatory treatment of credit risk. Within the Internal Ratings-Based approach (IRB)

Key words and phrases. The New Basel Capital Accord, copula, tail dependence, elliptical distribution, regular variation, non-parametric estimator, multivariate extreme value theory.

[15] the dependence structure, specifically the default correlation, of credit risky exposure represents a primary input for regulatory capital requirements. The IRB approach utilizes a single factor model for credit risk modelling. In particular, the asset portfolio is modelled by a multivariate normal distribution. However, multivariate normal distributions encounter two major insufficiencies leading to short comings in multivariate asset return modelling: On the one hand, their normal distributed margins are not flexible enough, and on the other hand, the copula related to the normal distribution (normal copula) does not possess the tail dependence property. This property enables the modelling of dependencies of extremal credit default events. To be more precise: The tail dependence concept describes the amount of dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a bivariate distribution. In this paper we propose substituting the normal copula by an elliptical copula which possesses the tail dependence property. Further we show that this property remains valid after a change of margins of the corresponding asset return random vector, i.e. the tail dependence concept is a copula property.

This paper is organized as follows: After an outline of the dependence structure modelling within The New Basel Capital Accord we present the theory of copulae as a general framework for modelling dependence. In section 3 we introduce the tail dependence concept as a copula property and give a characterization of tail dependence for elliptical copulae in section 4. Section 5 embeds the concepts of tail dependence and elliptical copulae into the framework of extreme value theory. In the last section we provide a parametric and non-parametric estimator for the tail dependence coefficient.

1. Credit risk modelling within The New Basel Capital Accord

The 1988 Basel Capital Accord is a current benchmark for many national regulatory laws related to "economic capital" on commercial bank lending businesses. This Accord requires banks to keep an 8% capital charge of the loan face value for any commercial loan in order to cushion losses from an eventual credit default. An increasing problem arises from the overall 8% capital charge which does not include the financial strength of the borrower and the value of the collateral. This led to an off-balance-sheet movement of low-risk credits and a retainment of high-risk credits. To overcome the insufficiency of credit-risk differentiation and other inadequacies, the Basel Committee launched the New Basel Capital Accord - Basel II which is expected to become national law in 2006. Basel II is divided into the Standard approach and the Internal Ratings-Based approach (IRB). Despite the Standard approach, which basically reflects the 1988 Basel Accord, the IRB approach calculates the "economic capital" by using a credit risk portfolio model. By credit-risk portfolio model we understand a function which maps a set of instrument-level and market-level parameters (cf. Gordy, [9], p. 1) to a distribution for portfolio credit losses over a specified horizon. In this context "economic capital" denotes the Value at Risk (VaR) of the portfolio loss distribution. Following the IRB-approach banks are now required to derive probabilities of default per loan (or exposure) via estimation or mapping. Together with other parameters the required regulatory capital increases with increasing probability of default and decreasing credit quality, respectively. Furthermore banks have an incentive to raise the "economic capital" in order to improve their own credit rating.

The IRB approach utilizes a single-factor model as credit-risk model which describes credit defaults by a two-state Merton model. This model can be compared to a simplified framework of CreditMetrics©. In particular, borrower i is linked to a random variable X_i which represents the normalized return of its assets, i.e.

$$X_i = \omega_i Z + \sqrt{1-\omega_i^2}\varepsilon_i, \qquad i = 1, \ldots, n, \tag{1.1}$$

where Z is a single common systematic risk factor related to all n borrowers and $\varepsilon_i,\ i = 1, \ldots, n$, denotes the borrower-specific risk. The random variables Z and $\varepsilon_i,\ i = 1, \ldots, n$, are assumed to be standard normally distributed and mutually independent. The parameters $\omega_i,\ i = 1, \ldots, n$, are called factor loadings and regulate the sensitivity towards the systematic risk factor Z.

The simplicity of the above single factor model has a significant advantage: It provides portfolio-invariant capital charges, i.e. the charges depend only on the loan's own properties and not on the corresponding portfolio properties. According to Gordy and Heitfeld [8], p. 5, this is essential for an IRB capital regime.

Observe that the IRB credit risk portfolio model (1.1) makes use of a multi-dimensional normal distribution, and thus the dependence structure of the portfolio's asset returns is that of a multi-dimensional normal distribution. Many empirical investigations ([10], [19], [6] and others) reject the normal distribution because of its inability to model dependence of extremal events. For instance, in the bivariate normal setting the probability that one component is large given that the other component is large tends to zero. In other words, the probability that the VaR in one component is exceeded given that the other component exceeds the VaR tends to zero. The concept of tail dependence, which we define soon, describes this kind of dependence for extremal events.

In the following we substitute the dependence structure of the above multivariate normal distribution by the dependence structure of elliptically contoured distributions, which contains the multi-dimensional normal distribution as a special case. This class, to be defined later, inherits most of the properties established in the IRB credit risk portfolio model. As additional advantage we can characterize dependence structures of elliptically contoured distributions which model dependencies of extremal default events.

Before continuing we clarify the general dependence concept of multivariate random vectors in the next section. Therefore the theory of copulae is needed.

2. The theory of copulae

The theory of copulae investigates the dependence structure of multi-dimensional random vectors. On the one hand, copulae are functions that join or "couple" multivariate distribution functions to their corresponding marginal distribution functions. On the other hand, a copula function itself is a multivariate distribution function with uniform margins on the interval $[0, 1]$. Copulae are of interest in credit-risk management for two reasons: First, as a way of studying the dependence structure of an asset portfolio irrespective of its marginal asset-return distributions; and second, as a starting point for constructing multi-dimensional distributions for asset portfolios, with a view to simulation. First we define the copula function in a common way (Joe [12], p. 12).

Definition 2.1. Let $C : [0,1]^n \to [0,1]$ be an n-dimensional distribution function on $[0,1]^n$. Then C is called a copula if it has uniformly distributed margins on the interval $[0,1]$.

The following theorem gives the foundation for a copula to inherit the dependence structure of a multi-dimensional distribution.

Theorem 2.2 (Sklar's theorem)**.** *Let F be an n-dimensional distribution function with margins $F_1, \ldots, F_n$. Then there exists a copula C, such that for all $x \in \mathbb{R}^n$*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{2.1}$$

If $F_1, \ldots, F_n$ are all continuous, then C is unique; otherwise C is uniquely determined on $RanF_1 \times \cdots \times RanF_n$. Conversely, if C is a copula and $F_1, \ldots, F_n$ are distribution functions, then the function F defined by (2.1) is an n-dimensional distribution function with margins $F_1, \ldots, F_n$.

We refer the reader to Sklar [22] or Nelsen [14] for the *proof.*

An immediate Corollary shows how one can obtain the copula of a multi-dimensional distribution function.

Corollary 2.3. *Let F be an n-dimensional continuous distribution function with margins $F_1, \ldots, F_n$. Then the corresponding copula C has representation*

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)), \quad 0 \le u_1, \ldots, u_n \le 1,$$

where $F_1^{-1}, \ldots, F_n^{-1}$ denote the generalized inverse distribution functions of $F_1, \ldots,$ F_n, i.e. for all $u_i \in (0,1) : F_i^{-1}(u_i) := \inf\{x \in \mathbb{R} \mid F_i(x) \ge u_i\},\ i = 1, \ldots, n.$

According to Schweizer and Wolff [21]: "... the copula is invariant while the margins may be changed at will, it follows that it is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strictly increasing transformations" and thus the copula function represents the dependence structure of a multivariate random vector. We add some more copula properties needed later.

Remarks.

(1) A copula is increasing in each component. In particular the partial derivatives $\partial C(u)/\partial u_i,\ i = 1 \ldots n$, exist almost everywhere.

(2) Consequently, the conditional distributions of the form

$$C(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n \mid u_j), \quad j = 1, \ldots, n, \tag{2.2}$$

exist.

(3) A copula C is uniformly continuous on $[0,1]^n$.

For more details regarding the theory of copulae we refer the reader to the monographs of Nelsen [14] and Joe [12].

3. Tail dependence: A copula property

Now we introduce the concepts of tail dependence and regularly varying (multivariate) functions. We will embed the tail dependence concept within the copula framework. Recall that multivariate distributions possessing the tail dependence property are of special practical interest within credit portfolio modelling, since they are able to incorporate dependencies of extremal credit default events. According to Hauksson et al. [10], Resnick [19], and Embrechts et al. [6] tail dependence plays an important role in extreme value theory, finance, and insurance models. Tail dependence models for multivariate distributions are mostly related to their bivariate marginal distributions. They reflect the limiting proportion of exceedence of one margin over a certain threshold given that the other margin has already exceeded that threshold. The following approach represents one of many possible definitions of tail dependence.

Definition 3.1 (Tail dependence, Joe [12], p. 33)**.** Let $X = (X_1, X_2)'$ be a 2-dimensional random vector. We say that X is *tail dependent* if

$$\lambda := \lim_{v\to 1^-} \mathbb{P}(X_1 > F_1^{-1}(v) \mid X_2 > F_2^{-1}(v)) > 0; \tag{3.1}$$

where the limit exists and F_1^{-1}, F_2^{-1} denote the generalized inverse distribution functions of X_1, X_2. Consequently, we say $X = (X_1, X_2)'$ is *tail independent* if λ equals 0. Further, we call λ the (upper) tail dependence coefficient.

Remark. Similarly, one may define the lower tail dependence coefficient by

$$\omega := \lim_{v\to 0^+} \mathbb{P}(X_1 \leq F_1^{-1}(v) \mid X_2 \leq F_2^{-1}(v)).$$

The following Proposition shows that tail dependence is a copula property. Thus many copula features translate to the tail dependence coefficient, for example the invariance under strictly increasing transformations of the margins.

Proposition 3.2. *Let X be a continuous bivariate random vector, then*

$$\lambda = \lim_{u\to 1^-} \frac{1 - 2u + C(u,u)}{1-u}, \tag{3.2}$$

where C denotes the copula of X. Analogous $\omega = \lim_{u\to 0^+} \frac{C(u,u)}{u}$ holds for the lower tail dependence coefficient.

Proof. Let F_1 and F_2 be the marginal distribution functions of X. Then

$$\begin{aligned}
\lambda &= \lim_{u\to 1^-} \mathbb{P}(X_1 > F_1^{-1}(u) \mid X_2 > F_2^{-1}(u)) \\
&= \lim_{u\to 1^-} \frac{\mathbb{P}(X_1 > F_1^{-1}(u), X_2 > F_2^{-1}(u))}{\mathbb{P}(X_2 > F_2^{-1}(u))} \\
&= \lim_{u\to 1^-} \frac{1 - F_2(F_2^{-1}(u)) - F_1(F_1^{-1}(u)) + C(F_1(F_1^{-1}(u)), F_2(F_2^{-1}(u)))}{1 - F_2(F_2^{-1}(u))} \\
&= \lim_{u\to 1^-} \frac{1 - 2u + C(u,u)}{1-u}.
\end{aligned}$$

□

Although we provided a simple characterization for upper and lower tail dependence by the last proposition, it will be still difficult and tedious to verify certain tail dependencies if the copula is not a closed-form expression, as in the case for most well-known elliptically contoured distributions. Therefore, the following Theorem gives another approach calculating tail dependence. We restrict ourselves to the upper tail.

Proposition 3.3. *Let X be a bivariate random vector with differentiable copula C. Then the (upper) tail dependence coefficient λ can be expressed using conditional* *probabilities if the following limit exists:*

$$\lambda = \lim_{v\to 1^-} \big(\mathbb{P}(U_1 > v|U_2 = v) + \mathbb{P}(U_2 > v|U_1 = v)\big), \tag{3.3}$$

where (U_1, U_2) are distributed according to the copula C of X.

Proof. Let C denote the copula of X which is assumed to be differentiable on the interval $(0,1)^2$. Therefore we may write $\mathbb{P}(U_1 \le v|U_2 = u) = \partial C(u,v)/\partial u$ and $\mathbb{P}(U_1 > v|U_2 = u) = 1 - \partial C(u,v)/\partial u$, respectively. The rule of L'Hospital implies that

$$\begin{aligned}\lambda = \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u} &= \lim_{u\to 1^-} \Big(-\Big(-2+\frac{dC(u,u)}{du}\Big)\Big)\\ &= \lim_{u\to 1^-} \Big(2 - \frac{\partial C(x,u)}{\partial x}\Big|_{x=u} - \frac{\partial C(u,y)}{\partial y}\Big|_{y=u}\Big)\\ &= \lim_{u\to 1^-} \big(\mathbb{P}(U_1 > u|U_2 = u) + \mathbb{P}(U_2 > u|U_1 = u)\big).\end{aligned}$$

□

Some of the following results for copulae of elliptically contoured distributions are characterized by regularly varying or O-regularly varying functions and multivariate regularly varying random vectors, which are defined as follows.

Definition 3.4. (Regular and O-regular variation of real valued functions)

1. A measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *regularly varying* (at ∞) with index $\alpha \in \mathbb{R}$ if for any $t > 0$
$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^\alpha.$$
2. A measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *O-regularly varying* (at ∞) if for any $t \ge 1$
$$0 < \liminf_{x\to\infty} \frac{f(tx)}{f(x)} \le \limsup_{x\to\infty} \frac{f(tx)}{f(x)} < \infty.$$

Thus, regularly varying functions behave asymptotically like power functions.

Definition 3.5 (Multivariate regular variation of random vectors)**.** An n-dimensional random vector $X = (X_1, \ldots, X_n)^T$ and its distribution are said to be regularly varying with limit measure ν if there exists a function $b(t) \nearrow \infty$ as $t \to \infty$ and a non-negative Radon measure $\nu \ne 0$ such that

$$t\mathbb{P}\Big(\Big(\frac{X_1}{b(t)}, \ldots, \frac{X_n}{b(t)}\Big) \in \cdot\Big) \xrightarrow{v} \nu(\cdot) \tag{3.4}$$

on the space $E = [-\infty,\infty]^n \backslash \{0\}$.

Notice that convergence $\xrightarrow{v}$ stands for vague convergence of measures, in the sense of Resnick [18], p. 140. It can be shown that (3.4) requires the existence of a constant $\alpha \ge 0$ such that for relatively compact sets $B \subset E$ (i.e. the closure $\overline{B}$ is compact in E)

$$\nu(tB) = t^{-\alpha}\nu(B), \quad t > 0. \tag{3.5}$$

Thus we say X is regularly varying with limit measure ν and index $\alpha \ge 0$, if (3.4) holds. Moreover, the function $b(\cdot)$ is necessarily regular varying with index $1/\alpha$. In the following we always assume $\alpha > 0$.

Later, when we consider elliptically contoured distributions, it turns out that polar coordinate transformations are a convenient way to deal with multivariate regular variation. Denote by $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : ||x|| = 1\}$ the $(n-1)$-dimensional unit sphere for some arbitrary norm $|| \cdot ||$ in $\mathbb{R}^n$. Then the polar coordinate transformation $T : \mathbb{R}^n \backslash \{0\} \to (0, \infty) \times \mathbb{S}^{n-1}$ is defined by

$$T(x) = \big(||x||, \frac{x}{||x||}\big) =: (r, a).$$

Observe, the point x can be seen as being distance r from the origin 0 away with direction $a \in \mathbb{S}^{n-1}$. It is well-known that T is a bijection with inverse transform $T^{-1} : (0,\infty) \times \mathbb{S}^{n-1} \to \mathbb{R}^n \backslash \{0\}$ given by $T^{-1}(r, a) = ra$. For notational convenience we denote the euclidian-norm by $||\cdot||_2$ and the related unit sphere by $\mathbb{S}_2^{n-1}$. The next proposition, stated essentially in Resnick [19], Proposition 2, characterizes multivariate regularly varying random vectors under polar-coordinate transformation.

Proposition 3.6. *The multivariate regular variation condition* (3.4) *is equivalent to the existence of a random vector* Θ *with values in the unit sphere* $\mathbb{S}^{n-1}$ *such that for all* $x > 0$

$$t\mathbb{P}\Big(\Big(\frac{||X||}{b(t)}, \frac{X}{||X||}\Big) \in \cdot\Big) \xrightarrow{v} c\nu_\alpha \mathbb{P}(\Theta \in \cdot), \quad as \quad t \to \infty, \tag{3.6}$$

where $c > 0$, ν_α *is a measure on Borel subsets of* $(0, \infty]$ *with* $\nu_\alpha((x, \infty]) = x^{-\alpha}$, $x > 0, \alpha > 0$, *and* $|| \cdot ||$ *denotes an arbitrary norm in* $\mathbb{R}^n$. *We call* $S(\cdot) := \mathbb{P}(\Theta \in \cdot)$ *the spectral measure of* X.

Remark. According to Stărică [23], p. 519, multivariate regular variation condition (3.4) is also equivalent to

$$\frac{\mathbb{P}(||X|| > tx, X/||X|| \in \cdot)}{\mathbb{P}(||X|| > t)} \xrightarrow{v} x^{-\alpha}\mathbb{P}(\Theta \in \cdot), \quad \text{as} \quad t \to \infty, \tag{3.7}$$

where $|| \cdot ||$ denotes an arbitrary norm in $\mathbb{R}^n$ and $S(\cdot) := \mathbb{P}(\Theta \in \cdot)$ is the *spectral measure* of X. Observe that regular variation of random variables is equivalent to regular variation of its distribution's tail function.

Notice that the latter proposition also implies that the multivariate regular variation property (3.6) does not depend on the choice of the norm. For more details regarding regular variation, O-regular variation, and multivariate regular variation we refer the reader to Bingham, Goldie, and Teugels [1], pp. 16, pp. 61, and pp. 193 and Resnick [18], pp. 12, pp. 250.

4. Tail dependence of elliptical copulae

Elliptically contoured distributions (in short: elliptical distributions) play a significant role in risk management due to many properties which fit very well in the Value at Risk and Markowitz framework. The best known elliptical distributions are the multivariate normal distribution, the multivariate t-distribution, the multivariate logistic distribution, and multivariate symmetric general hyperbolic distribution.

Definition 4.1 (Elliptical distribution)**.** Let X be an n-dimensional random vector and $\Sigma \in \mathbb{R}^{n\times n}$ be a symmetric positive semi-definite matrix. If $X - \mu$, for some $\mu \in \mathbb{R}^n$, has a characteristic function of the form $\phi_{X-\mu}(t) = \Phi(t^T\Sigma t)$, then X is said to be elliptically distributed with parameters μ, Σ, and Φ. Let $E_n(\mu, \Sigma, \Phi)$

denote the class of elliptically contoured distributions with the latter parameters. We call Φ the *characteristic generator*.

Definition 4.2 (Elliptical copulae)**.** We say C is an elliptical copula, if it is the copula of an elliptically contoured distribution.

Remark. The density function, if it exists, of an elliptically contoured distribution has the following form:

$$f(x) = |\Sigma|^{-1/2} g((x-\mu)^T \Sigma^{-1}(x-\mu)), \quad x \in \mathbb{R}^n, \tag{4.1}$$

for some function $g : \mathbb{R} \to \mathbb{R}_+$, which we call the *density generator*. Observe that the name "elliptically contoured" distribution is related to the elliptical contours of the latter density.

Examples. In the following we give some examples of density generators for n-dimensional elliptical distributions. Here C_n denotes a normalizing constant depending only on the dimension n.

1. Normal distribution: $g(u) = C_n \exp(-u/2)$.
2. t-distribution: $g(u) = C_n(1 + \frac{t}{m})^{-(n+m)/2}, m \in \mathbb{N}$.
3. logistic distribution: $g(u) = C_n \exp(-u)/(1 + \exp(-u))^2$.
4. Symmetric generalized hyperbolic distribution: $g(u) = C_n K_{\lambda - \frac{n}{2}}(\sqrt{\psi(\chi + u)})/(\sqrt{\chi + u})^{\frac{n}{2} - \lambda}$, $u > 0$, where $\psi, \chi > 0$, $\lambda \in \mathbb{R}$, and K_ν denotes the modified Bessel function of the third kind (or Macdonald function).

A characteristic property of elliptical distributions is that all margins are elliptically distributed with the same characteristic generator or density generator, respectively. However, in most risk models one encounters the problem that the margins of the multivariate asset-return random vector are empirically not of the same distribution type. As a solution we propose joining appropriate marginal distributions with an elliptical copula, because of its well-known and statistically tractable properties. One important issue is of course the estimation of the copula or the copula parameters, respectively. According to Theorem 2.15 in Fang, Kotz, and Ng [7], elliptical copulae C corresponding to elliptically distributed random vectors $X \in E_n(\mu, \Sigma, \Phi)$ with positive-definite matrix Σ, are uniquely determined up to a positive constant by the matrix $R_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$, $1 \le i, j \le n$ and the characteristic generator Φ or the density generator g, respectively. Uniqueness is obtained by setting $|\Sigma| = 1$ without loss of generality. Observe that R corresponds to the linear correlation matrix, if it exists. Embrechts et al. [6] propose the following robust estimator for R via Kendall's Tau τ. This estimator is based on the relationship $\tau(X_i, X_j) = \frac{2}{\pi}\arcsin(R_{ij})$, $1 \le i, j \le n$, for X_i and X_j having continuous distributions:

$$\hat{R} = \sin(\pi\hat{\tau}/2) \qquad \text{with} \quad \hat{\tau} = \frac{c-d}{\binom{n}{2}}, \tag{4.2}$$

where c and d denote the number of concordant and discordant tuples of a bivariate random sample. The characteristic generator Φ or the density generator g can be estimated via non-parametric estimators as they are discussed in Bingham and Kiesel [2].

An immediate question arises: Which elliptical copula should one choose? The previous discussion showed that tail dependence represents a desired copula property in the context of credit-risk management. It is well-known that the Gaussian copula with correlation coefficient $\rho < 1$ does not inherit tail dependence (see Schmidt [20] for more details). Therefore we seek a characterization of elliptical copulae possessing the latter property. The following stochastic representation turns out to be very useful.

Let X be an n-dimensional elliptically distributed random vector, i.e. $X \in E_n(\mu, \Sigma, \Phi)$, with parameters μ and symmetric positive semi-definite matrix Σ, $\text{rank}(\Sigma) = m,\ m \leq n$. Then

$$X \overset{d}{=} \mu + R_m A' U^{(m)}, \tag{4.3}$$

where $A'A = \Sigma$ and the univariate random variable $R_m \geq 0$ is independent of the m-dimensional random vector $U^{(m)}$. The random vector $U^{(m)}$ is uniformly distributed on the unit sphere $\mathbb{S}_2^{n-1}$ in $I\!R^m$. In detail, R_m represents a radial part and $U^{(m)}$ represents an angle of the corresponding elliptical random vector X. We call R_m the *generating variate* of X. The above representation is also applicable for fast simulation of multi-dimensional elliptical distributions and copulae. Especially in risk-management practice, where large exposure portfolios imply high-dimensional distributions, one is interested in fast simulation technics.

Although tail dependence is a copula feature we will state, for the purpose of generality, the next characterization of tail dependence for elliptical distributions. For this we need the following condition, which is easy to check in the context of density generators.

Condition 4.3. *Let $h : I\!R^+ \to I\!R^+$ be a measurable function eventually decreasing such that for some $\varepsilon > 0$*

$$\limsup_{x \to \infty} \frac{h(tx)}{h(x)} \leq 1 - \varepsilon \qquad \text{uniformly} \; \forall \; t > 1.$$

Theorem 4.4. *Let $X \in E_n(\mu, \Sigma, \Phi),\ n \geq 2$, with positive-definite matrix Σ. If X possesses a density generator g then*
α) all bivariate margins of X possess the tail dependence property if g is regularly varying, and
β) if X possesses a tail dependent bivariate margin and g satisfies Condition 4.3, then g must be O-regularly varying.

The *Proof* and examples are given in Schmidt [20].

Remark. Although we cannot show the equivalence of tail dependence and regularly varying density generator, most well-known elliptical distributions and elliptical copulae are given either by a regularly varying or a not O-regularly varying density generator. That justifies a restriction to the class of elliptical copulae with regularly varying density generator if one wants to incorporate tail dependence.

Additionally we can state a closed form expression for the tail dependence coefficient of an elliptically contoured random vector $(X_1, X_2)' \in E_2(\mu, \Sigma, \Phi)$ with positive-definite matrix Σ, and the corresponding elliptical copula, having a regular varying density generator g with index $-\alpha/2 - 1 < 0:$

$$\lambda = \lambda(\alpha, \rho) = \frac{\displaystyle\int_0^{h(\rho)} \frac{u^\alpha}{\sqrt{1-u^2}} du}{\displaystyle\int_0^1 \frac{u^\alpha}{\sqrt{1-u^2}} du}, \tag{4.4}$$

with $\rho := \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$ and $h(\rho) := \left(1 + \frac{(1-\rho)^2}{1-\rho^2}\right)^{-1/2}$ (see also Figure 1). This formula has been developed in the proof of Theorem 5.2 in Schmidt [20], p. 20. Note, that ρ corresponds to the correlation coefficient when this exists (see Fang, Kotz, and Ng [7], p. 44, for the covariance formula of elliptically contoured distributions). We remark that the (upper) tail dependence coefficient λ coincides with the lower tail dependence coefficient and depends only on the "correlation" coefficient ρ and the regular variation index α.

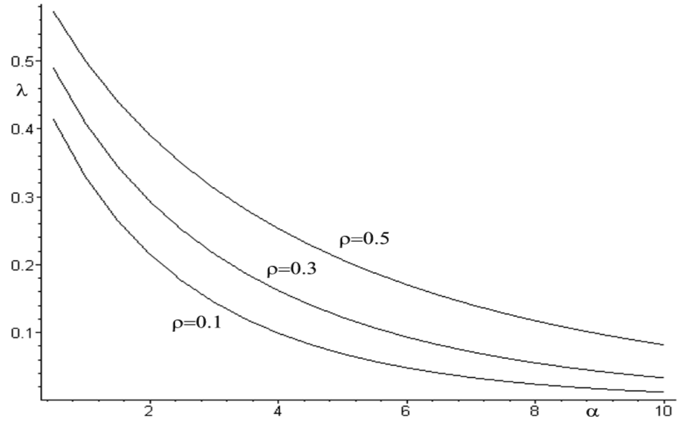

FIGURE 1. Tail dependence coefficient λ versus regular variation index α for $\rho = 0.5,\ 0.3,\ 0.1$

For completeness we state the following generalization of Theorem 4.4. The proof can also be found in Schmidt [20].

Theorem 4.5. *Let* $X \in E_n(\mu, \Sigma, \Phi)$, $n \geq 2$, *with positive-definite matrix* Σ *and stochastic representation* $X \stackrel{d}{=} \mu + R_n A' U^{(n)}$. *Denote by* H_n *the distribution function of* R_n.
α) *If* X *has a tail dependent bivariate margin, then the tail function* $1 - H_n$ *of* R_n *must be O-regularly varying.*
β) *If* X *has a tail dependent bivariate margin, then the tail function* $1 - F_i$ *must be O-regularly varying, where* F_i *denote the distribution functions of the univariate margins of* X_i, $i = 1, \ldots, n$.
γ) *Suppose the distribution function* H_n *of* R_n *has a regularly varying tail. Then all bivariate margins are tail dependent.*

5. Tail dependence: A tool of multivariate extreme value theory

In this section we embed the concepts of tail dependence and elliptical copulae, introduced in section 3 and 4, into the framework of multivariate extreme value theory. Extreme value theory is the natural choice for inferences on extremal events of random vectors or the tail behavior of probability distributions. Usually one approximates the tail of a probability distribution by an appropriate extreme value distribution. In the one-dimensional setting the class of extreme value distributions has a solely parametric representation, so it suffices to apply parametric estimation methods. By contrast, multi-dimensional extreme value distributions are characterized by a parametric and a non-parametric component. This leads to more complicated estimation methods. First we provide the necessary background for our purpose. Let $X,\ X^{(1)},\ X^{(2)},\dots,X^{(m)},\ m\in\mathbb{N}$ be independent multivariate random vectors with common continuous distribution function F. We say X or its distribution is in the domain of attraction of a multivariate extreme value distribution G if there exists a sequence of normalizing constants $(a_{mi})_{m=1}^{\infty}$, $(b_{mi})_{m=1}^{\infty}$ with $a_{mi}>0$ and $b_{mi}\in\mathbb{R},\ i=1,\dots,n$ such that

$$\mathbb{P}\Big(\frac{\max_{1\le i\le m}X_1^{(m)}-b_{m1}}{a_{m1}}\le x_1,\ \dots\ ,\frac{\max_{1\le i\le m}X_n^{(m)}-b_{mn}}{a_{mn}}\le x_n\Big) \tag{5.1}$$

converges to the limit distribution function G with non-degenerate margins as $m\to\infty$. In particular, the latter is equivalent to

$$\lim_{m\to\infty}F^m(a_{m1}x_1+b_{m1},\dots,a_{mn}x_n+b_{mn})=G(x_1,\dots,x_n).$$

Before turning to elliptical copulae we prove the following theorem for elliptical distributions.

Theorem 5.1. *Let $X\in E_n(\mu,\Sigma,\Phi)$ with stochastic representation $X\overset{d}{=}\mu+R_nA'U^{(n)}$ and positive-definite matrix Σ. If the generating variate R_n possesses a regularly varying tail function, then X lies in the domain of attraction of an extreme value distribution.*

Proof. Suppose $X\in E_n(\mu,\Sigma,\Phi)$ with stochastic representation $X\overset{d}{=}\mu+R_nA'U^{(n)}$ and positive-definite matrix Σ. We start by showing that a regularly varying R_n requires X to be in the class of multivariate regularly varying random vectors, introduced in Definition 3.5. Consider first the case $\mu=0$ and $\Sigma=I$, i.e. $X\overset{d}{=}R_nU^{(n)}$. We need the following characterization of vague convergence stated in Resnick [18], Proposition 3.12, p. 142: A sequence of Radon measures ν_m on some space $\mathbb{E}$ converges vaguely to a Radon measure ν on $\mathbb{E}$ if and only if $\lim_{m\to\infty}\nu_m(B)=\nu(B)$ for all relatively compact Borel sets $B\in\mathbb{E}$ (i.e. the closure $\overline{B}$ is compact in $\mathbb{E}$) with $\nu(\partial B)=0$. Note that the Borel sets $(x,\infty]\times C,\ x>0$, of $(0,\infty]\times\mathbb{S}_2^{n-1}$ represent a generating Π-system of the class of relatively compact sets of $(0,\infty]\times\mathbb{S}_2^{n-1}$. Thus it suffices to consider for $x>0$ and $0<b(t)\nearrow\infty$

$$\begin{aligned}t\mathbb{P}\Big(T\Big(\frac{X}{b(t)}\Big)\in(x,\infty]\times C\Big)&=t\mathbb{P}\Big(\Big(\frac{||X||_2}{b(t)},\frac{X}{||X||_2}\Big)\in(x,\infty]\times C\Big)\\&=t\mathbb{P}\Big(\frac{R_n}{b(t)}>x\Big)\mathbb{P}(U^{(n)}\in C)\to x^{-\alpha}\mathbb{P}(U^{(n)}\in C)\quad\text{as } t\to\infty,\end{aligned}$$

since R_n and $U^{(n)}$ are stochastically independent and R_n is regularly varying. Further the latter convergence is locally uniformly due to locally uniform convergence of regularly varying functions (see Resnick [18], Proposition 0.5, p. 17) and the absolute continuity of $U^{(n)}$. Applying Proposition 3.6 yields the existence of $b(t)\nearrow\infty$ with

$$t\mathbb{P}\Big(\frac{X}{b(t)}\in\cdot\Big)\overset{v}{\to}\nu(\cdot) \tag{5.2}$$

for a Radon measure ν on $E=[-\infty,\infty]^n\backslash\{0\}$and locally uniform convergence transfers, because T and its inverse are continuous functions on E.

Let now $\mu \in I\!R^n$ and $\Sigma \in I\!R^{n\times n}$, positive-definite, be arbitrary. Set $A \in I\!R^{n\times n}$ such that $\Sigma = A'A$ and A is a regular matrix. Define the Radon measure $\rho(\cdot) := \nu(A'\cdot)$ on E. Then

$$t I\!P\Big(\frac{X}{b(t)} \in \cdot\Big) = t I\!P\Big(\frac{A'R_nU^{(n)}}{b(t)} + \frac{\mu}{b(t)} \in \cdot\Big) \overset{v}{\rightarrow} \rho(\cdot) \tag{5.3}$$

holds because of the locally uniform convergence property. Further the sets $A'B := \{A'x \mid x \in \text{B relatively compact in } E\}$ are relatively compact on E and $\nu(\partial(A'B)) = 0$ if $\rho(\partial B) = \nu(A'\partial B) = 0$, since $\partial(A'B) \subset A'(\partial B)$ holds for regular matrixes A' ($(A')^{-1}x$ is a continuous function on E). Again Proposition 3.6 yields

$$t I\!P\Big(\frac{||X||}{b(t)} > x, X/||X|| \in \cdot\Big) \overset{v}{\rightarrow} cx^{-\alpha} I\!P(\Theta \in \cdot),\ c > 0, \tag{5.4}$$

for some spectral measure $S(\cdot) = I\!P(\Theta \in \cdot)$ on the unit sphere $\mathbb{S}_2^{n-1}$. We refer the reader to Hult and Lindskog [11] for explicit calculations of the spectral measure with respect to the euclidian and the max-norm. Finally (5.4) and Corollary 5.18 in Resnick [18], p. 281, require X to be in the domain of attraction of an extreme value distribution □

Typically, elliptically contoured distributions are given by their density function or their density generator, respectively. Thus, the next Corollary turns out to be helpful.

Corollary 5.2. *Let $X \in E_n(\mu, \Sigma, \Phi)$ be an elliptically contoured distribution with regularly varying density generator g introduced in (4.1). Then X is in the domain of attraction of an extreme value distribution.*

Proof. According to Proposition 3.7 in Schmidt [20], p. 10, a regularly varying density generator implies a regularly varying density function of the generating variate R_n. In particular the latter proposition yields the existence of a density function of R_n. By Karamata's Theorem (see Bingham, Goldie, and Teugels [1], p. 26) regular variation is transferred to the tail function of R_n. The corollary follows by Theorem 5.1. □

The following calculation clarifies the relationship between the spectral measure arising from multivariate regular variation of random vectors (see (3.6)) and extreme value distributions. According to Corollary 5.18 in Resnick [18], p. 281, every multivariate regularly varying random vector with associated spectral measure $S(\cdot)$ is in the domain of attraction of an extreme value distribution G with representation

$$G(x) = \exp\big(-\nu([-\infty, x]^c)\big), \quad x \in I\!R_+^n, \tag{5.5}$$

where $\nu(\{x \in I\!R^n\backslash\{0\} :\ ||x|| > t, x/||x|| \in \cdot\}) = t^{-\alpha}S(\cdot)$ and $[-\infty, x]^c := \{y \in E \mid y_i > x \text{ for some } i = 1, \dots, n\}$. In the literature ν is referred to as the *exponent measure* and ν_T is the corresponding measure under polar coordinate transformation $T(x) = (||x||, x/||x||)$. In particular ν_T represents a product measure and $S(\cdot) = \nu(\{x \in I\!R^n\backslash\{0\} :\ ||x|| > 1, x/||x|| \in \cdot\})$. Recall that T is a bijection on E

and $T^{-1}(r,a) = ra$ holds. Set $\overline{\mathbb{S}}^{n-1} := \mathbb{S}^{n-1} \backslash (-\infty, 0]^n$. Then for $x \in I\!R_+^n$

$$\begin{aligned}
\nu([-\infty, x]^c) &= \nu_T(T([-\infty, x]^c)) \\
&= \nu_T(\{(r,a) \in I\!R_+ \times \mathbb{S}^{n-1} \mid ra_i > x_i \text{ for some } i = 1, \dots, n\}) \\
&= \nu_T((r,a) \in I\!R_+ \times \mathbb{S}^{n-1} \mid a \in \overline{\mathbb{S}}^{n-1},\ r > \min\{\frac{x_i}{a_i},\ i \in I_a\} =: g(a)) \\
&= \int_{\overline{\mathbb{S}}^{n-1}} \int_{g(a)}^{\infty} \frac{1}{\alpha+1} \frac{1}{r^{\alpha+1}}\, dr\, S(da) = \int_{\overline{\mathbb{S}}^{n-1}} \frac{1}{g(a)^\alpha}\, S(da) \\
&= \int_{\overline{\mathbb{S}}^{n-1}} \frac{1}{\big[\min\big\{\frac{x_i}{a_i}, i \in I_a\big\}\big]^\alpha}\, S(da) \\
&= \int_{\overline{\mathbb{S}}^{n-1}} \big[\max\big\{\frac{a_i}{x_i}, i \in I_a\big\}\big]^\alpha\, S(da),
\end{aligned}$$

where $I_a = \{j \in \{1, \dots, n\} \mid a_j > 0\}$. We summarize the above results in the following proposition.

Proposition 5.3. *Let X be a multivariate regularly varying random vector according to Definition 3.5. Then X is in the domain of attraction of a multi-dimensional extreme value distribution*

$$G(x) = \exp\Big(-\int_{\overline{\mathbb{S}}^{n-1}} \big[\max\big\{\frac{a_i}{x_i}, i \in I_a\big\}\big]^\alpha\, S(da)\Big), \quad x \in I\!R_+^n, \tag{5.6}$$

with spectral measure $S(\cdot)$ living on the unit sphere $\mathbb{S}^{n-1}$.

In general, multi-dimensional extreme value distributions are characterized by an extreme value index and a finite measure, which is commonly referred to as the spectral or angular measure. According to the latter proposition, a multivariate regularly varying random vector is in the domain of attraction of an extreme value distribution with spectral measure coinciding with that of Definition 3.5.

For the family of elliptically contoured distributions the spectral measure is given in closed form. Especially for an elliptical random vector $X \in E_n(0, I, \Phi)$ with stochastic representation $X \overset{d}{=} R_n U^{(n)}$ and R_n having a regularly varying tail function with index α we obtain

$$G(x) = \exp\Big(-\int_{\overline{\mathbb{S}}_2^{n-1}} \big[\max\big\{\frac{a_i}{x_i}, i \in I_a\big\}\big]^\alpha\, da\Big), \quad x \in I\!R_+^n, \tag{5.7}$$

with $\mathbb{S}_2^{n-1}$ denoting the unit sphere with respect to the euclidian norm $||\cdot||_2$. Recall that the spectral measure $S(\cdot)$ is uniformly distributed on the unit sphere $\mathbb{S}_2^{n-1}$. Moreover, in the bivariate setup straightforward calculation yields

$$\begin{aligned}
G(x_1, x_2) = \exp\Big(-\frac{1}{2\pi}\Big(&\frac{\sqrt{\pi}}{2}\frac{\Gamma((1+\alpha)/2)}{\Gamma(1+\alpha/2)}\big(\frac{1}{x_1^\alpha} + \frac{1}{x_2^\alpha}\big) \\
&+ \frac{1}{x_1^\alpha}\int_0^{\tan^{-1}(x_2/x_1)} \cos^\alpha\theta\, d\theta + \frac{1}{x_2^\alpha}\int_{\tan^{-1}(x_2/x_1)}^{\pi/2} \sin^\alpha\theta\, d\theta\Big)\Big).
\end{aligned} \tag{5.8}$$

Having established the connection among elliptically contoured distributions and multivariate extreme value theory, we now turn towards the relationship between the tail dependence coefficient, elliptical copulae, and bivariate extreme value theory. In the following we will always assume that the bivariate random vector X

is in the domain of attraction of an extreme value distribution. Recall, for a bivariate random vector X with distribution function F in the domain of attraction of an extreme value distribution G there must exist constants $a_{mi} > 0$ and $b_{mi} \in \mathbb{R},\ i = 1, 2$, such that

$$\lim_{m\to\infty} F^m(a_{m1}x_1 + b_{m1}, a_{m2}x_n + b_{m2}) = G(x_1, x_2).$$

Transformation of the margins of G to so-called standard Frechét margins yields

$$\lim_{m\to\infty} F_*^m(mx_1, mx_2) = G^*(x_1, x_2), \tag{5.9}$$

where F_* and G^* are the standardized distributions of F and G, respectively, i.e. $G_{*i}(x_i) = \exp(-1/x_i),\ x_i > 0,\ i = 1, 2$, and

$$F_*(x_1, x_2) = F\Big(\Big(\frac{1}{1-F_1}\Big)^{-1}(x_1), \Big(\frac{1}{1-F_2}\Big)^{-1}(x_2)\Big).$$

This standardization does not introduce difficulties as shown in Resnick [18], Proposition 5.10, p. 265. Moreover the following continuous version of (5.9) can be shown:

$$\lim_{t\to\infty} F_*^t(tx_1, tx_2) = G^*(x_1, x_2), \tag{5.10}$$

or equivalently

$$\lim_{t\to\infty} t(1 - F_*(tx_1, tx_2)) = -\log(G^*(x_1, x_2)). \tag{5.11}$$

Summarizing the above facts, we obtain

$$\begin{aligned}
&\lim_{t\to\infty} t\Big(1 - F\Big(\Big(\frac{1}{1-F_1}\Big)^{-1}(tx_1), \Big(\frac{1}{1-F_2}\Big)^{-1}(tx_2)\Big)\Big)\\
&\quad = -\log G\Big(\Big(\frac{1}{-\log G_1}\Big)^{-1}(x_1), \Big(\frac{1}{-\log G_2}\Big)^{-1}(x_2)\Big)\\
&\quad = -\log G^*(x_1, x_2). \qquad (5.12)
\end{aligned}$$

Thus the tail dependence coefficient, if it exists, can be expressed as

$$\begin{aligned}
\lambda &= \lim_{v\to 1^-} \mathbb{P}(X_1 > F_1^{-1}(v) \mid X_2 > F_2^{-1}(v))\\
&= \lim_{v\to 1^-} \mathbb{P}(X_1 > F_1^{-1}(v), X_2 > F_2^{-1}(v))/(1-v)\\
&= \lim_{t\to\infty} t\mathbb{P}\big(X_1 > F_1^{-1}\big(1 - \frac{1}{t}\big), X_2 > F_2^{-1}\big(1 - \frac{1}{t}\big)\big).
\end{aligned}$$

Further, easy calculation shows that

$$\begin{aligned}
-\log G^*(x_1, x_2) &= \lim_{t\to\infty} t\Big(1 - \mathbb{P}\Big(X_1 \le \Big(\frac{1}{1-F_1}\Big)^{-1}(tx_1), X_2 \le \Big(\frac{1}{1-F_2}\Big)^{-1}(tx_2)\Big)\Big)\\
&= \frac{1}{x_1} + \frac{1}{x_2} - \lim_{t\to\infty} \mathbb{P}\Big(X_1 > F_1^{-1}\Big(1 - \frac{1}{tx_1}\Big), X_2 > F_2^{-1}\Big(1 - \frac{1}{tx_2}\Big)\Big)
\end{aligned}$$

and hence

$$\lambda = 2 + \log G\Big(\Big(\frac{1}{-\log G_1}\Big)^{-1}(1), \Big(\frac{1}{-\log G_2}\Big)^{-1}(1)\Big). \tag{5.13}$$

The latter equation shows how one could model the tail dependence coefficient by choosing an appropriate bivariate extreme value distribution.

Using the above results, for an elliptical random vector $X \in E_n(0, I, \Phi)$ with stochastic representation $X \stackrel{d}{=} R_n U^{(n)}$ and R_n having a regularly varying tail function with index α, we derive

$$\lambda = \frac{\int_{\pi/4}^{\pi/2} \cos^\alpha \theta \, d\theta}{\int_0^{\pi/2} \cos^\alpha \theta \, d\theta}. \tag{5.14}$$

Observe that for $X \in E_n(0, I, \Phi)$ this formula coincides with formula (4.4) after a standard substitution.

Within the framework of copulae we can rewrite (3.2) and (5.13) to obtain

$$\begin{aligned} \lambda &= 2 - \lim_{t \to \infty} t \Big(1 - C\Big(1 - \frac{1}{t}, 1 - \frac{1}{t}\Big)\Big) \\ &= 2 + \log\Big(C_G\Big(\frac{1}{e}, \frac{1}{e}\Big)\Big), \end{aligned} \tag{5.15}$$

where C and C_G denote the copula of F and G, respectively. Using the notation of co-copulae (see Nelsen [14], p. 29) (5.15) implies

$$\lambda = 2 - \lim_{t \to \infty} t \, C_{co}\Big(\frac{1}{t}, \frac{1}{t}\Big) = 2 + \log\Big(C_G\Big(\frac{1}{e}, \frac{1}{e}\Big)\Big). \tag{5.16}$$

The above results lead to the observation that a bivariate random vector inherits the tail dependence property if the standardized distribution F_* or the related copula function C (equals the copula of F) lies in the domain of attraction of an extreme value distribution which does not have independent margins. Consequently it is not necessary that the bivariate distribution function F itself is in the domain of attraction of some extreme value distribution. This is an important property for asset portfolio modelling.

Remark. In particular every bivariate regularly varying random vector with a spectral measure not concentrated on $(c, 0)^T$ and $(0, c)^T$ for some $c > 0$ possesses the tail dependence property according to Corollary 5.25 in Resnick [18], p. 292. This is in line with Theorem 4.4 and Theorem 4.5, since the spectral measure of a non-degenerated elliptical distribution is not concentrated on single points.

Based on the previous results we finish this section with an important theorem about elliptical copulae.

Theorem 5.4. *Let C be an elliptical copula corresponding to an elliptical random vector $X \stackrel{d}{=} \mu + R_n A' U^{(n)} \in E_n(\mu, \Sigma, \Phi)$, Σ positive-definite, with regularly varying generating variate R_n or regularly varying density generator. Then C is in the domain of attraction of some extreme value distribution and all bivariate margins of C possess the tail dependence property.*

Proof. Suppose $X \stackrel{d}{=} \mu + R_n A' U^{(n)} \in E_n(\mu, \Sigma, \Phi)$, Σ positive-definite, with regularly varying generating variate R_n or regularly varying density generator. Then Theorem 5.1 and Corollary 5.2 require X to be in the domain of attraction of some extreme value distribution. According to Proposition 5.10 in Resnick [18], p. 265, all margins $X_i,\ i = 1, \ldots, n$ and the standardized distribution F_*, i.e. $F_*(x_1, \ldots, x_n) = F((1/(1 - F_1))^{-1}(x_1), \ldots, (1/(1 - F_n))^{-1}(x_n)$ are in the domain of attraction of an extreme value distributions and $\lim_{m \to \infty} F_*^m(mx_1, \ldots, mx_n) = G(x_1, \ldots, x_n)$. Since $F_*(x_1, \ldots, x_n) = C(1 - 1/x_1, \ldots, 1 - 1/x_n) = C_*(x_1, \ldots, x_n),\ x_1, \ldots, x_n \geq 1$

and uniform distributions on $[0,1]$ are in the domain of attraction of some extreme value distribution, we conclude again with Proposition 5.10 in Resnick [18] that C is in the domain of attraction of an extreme value distribution. Applying Proposition 3.1 in Schmidt [20], p.7, every bivariate margin of C is an elliptical copula with regularly varying generating variate or regularly varying density generator. Thus tail dependence for all bivariate margins of C follows by the results stated above this theorem. □

6. Estimating the tail dependence coefficient for elliptical copulae

Suppose $X,\ X^{(1)},\dots,X^{(m)}$ are iid bivariate random vectors with distribution function F and elliptical copula C. Further we assume continuous marginal distribution functions $F_i,\ i=1,2$. There are several parametric and non-parametric estimation methodologies for the tail dependence coefficient of an elliptical copula available. We distinguish between two kinds of bivariate random vectors possessing an elliptical copula as dependence structure: Those which are elliptically distributed and those which are not elliptically distributed, i.e. the margins might follow different distributions. Statistics testing for tail dependence and tail independence are given in Ledford and Tawn [13].

At the presence of tail dependence Theorem 4.4 and Theorem 4.5 justify to consider only elliptical copulae with regularly varying generating variate or regularly varying density generator.

i) First, we consider the case of an elliptically contoured bivariate random vector X. In particular we consider an elliptical random vector $X \stackrel{d}{=} \mu + R_2 A' U^{(2)}$ with regularly varying generating variate R_2, i.e. for $x>0$

$$\lim_{t\to\infty}\frac{\mathbb{P}(||X||_2>tx)}{\mathbb{P}(||X||_2>t)}=\lim_{t\to\infty}\frac{\mathbb{P}(R_2>tx)}{\mathbb{P}(R_2>t)}=x^{-\alpha},\ \alpha>0.$$

Then formula (4.4) shows that the tail dependence coefficient λ depends only on the tail index α and the "correlation" coefficient ρ, precisely $\lambda=\lambda(\alpha,\rho)$. A robust estimator $\hat\rho$ for ρ based on its relationship to Kendall's Tau was given in (4.2). Regarding the tail index α there are several well-known estimators obtainable from extreme value literature. Among these, the Hill estimator represents a natural one for the tail index α :

$$\hat\alpha_m=\Big(\frac{1}{k}\sum_{j=1}^{k}\log||X_{(j,m)}||_2-\log||X_{(k,m)}||_2\Big)^{-1}, \tag{6.1}$$

where $||X_{(j,m)}||_2$ denotes the j-th order statistics of $||X^{(1)}||_2,\dots,||X^{(m)}||_2$ and $k=k(m)\to\infty$ is chosen in an appropriate way; for a discussion on the right choice we refer the reader to Embrechts et al. [5], pp. 341.

ii) Now we consider the case that X is not elliptically distributed but is still in the domain of attraction of some extreme value distribution. Then we can estimate the tail dependence coefficient using the homogeneity property (3.5) and the spectral measure representation (5.6) arising from the limiting extreme value distribution. Einmahl et al. [3] and Einmahl et al. [4] propose a non-parametric and a semi-parametric estimator for the spectral measure of an extreme value distribution.

iii) Finally, if we have to reject that X follows an elliptical distribution and X is in the domain of attraction of an extreme value distribution, we propose the following estimator for λ which is based on the copula representation (5.15). Let

C_m be the empirical copula defined by

$$C_m(u_1,u_2)=F_m(F_{1m}^{-1}(u_1),F_{2m}^{-1}(u_2)), \tag{6.2}$$

with F_m, F_{im} denoting the empirical distribution functions corresponding to F, F_i, $i=1,2$. Let $R_{m1}^{(j)}$ and $R_{m2}^{(j)}$ be the rank of $X_1^{(j)}$ and $X_2^{(j)}$, $j=1,\dots,m$, respectively. Then

$$\begin{aligned}\hat\lambda_m &= 2-\frac{m}{k}\Big(1-C_m\Big(1-\frac{k}{m},1-\frac{k}{m}\Big)\Big)\\ &= 2-\frac{m}{k}+\frac{1}{k}\sum_{j=1}^{m}\mathbf{1}_{\{R_{m1}^{(j)}\le m-k,R_{m2}^{(j)}\le m-k\}}\end{aligned} \tag{6.3}$$

with $k=k(m)\to\infty$ and $k/m\to 0$ as $m\to\infty$. The optimal choice of k is related to the usual variance-bias problem, which we address in a forthcoming work. The next theorem states the strong consistency property of $\hat\lambda_m$.

Theorem 6.1. *Let X be a bivariate random vector with elliptical copula C having a regularly varying density generator or regularly varying generating variate. Let $\hat\lambda_m$ be the tail dependence coefficient estimator given in (6.3). If $k=k(m)\to\infty$, $k/m\to 0$, $k/\log(\log m)\to\infty$ as $m\to\infty$ then*

$$\hat\lambda_m\to\lambda \quad \textit{almost surely} \quad \textit{as} \quad m\to\infty.$$

Proof. Let X possess an elliptical copula C with regularly varying density generator or regularly varying generating variate. According to Stute [24], p. 371, the distribution of C_m in (6.2) does not dependent on the marginal distributions F_1 and F_2 such that w.l.o.g we may assume that F_i, $i=1,2$, are uniform distributions on the unit interval and we are in the copula framework. Theorem 5.4 yields that C is in the domain of attraction of an extreme value distribution. The strong consistence is now a special case of Theorem 1.1 in Qi [17] because of the uniform convergence of $C^m(1-1/(mx_1),\dots,1-1/(mx_n))$ to its corresponding extreme value distribution. □

Asymptotic normality will be addressed in a forthcoming work.

The figures below graphically summarize the tail dependence properties of four financial data-sets. We provide the scatter plots of daily negative log-returns of the financial securities and compare them to the corresponding tail dependence coefficient estimate (6.3) for various k. Both plots give an intuition for the presence of tail dependence and the order of magnitude of the tail dependence coefficient. For modelling reasons we assume that that the daily log-returns are iid observations. All plots related to the estimation of the tail dependence coefficient show the typical variance-bias problem for various k. In particular, a small k comes along with a large variance of the estimate, whereas an increasing k results in a strong bias. In the presence of tail dependence, k is chosen such that the tail dependence coefficient estimate $\hat\lambda$ lies on a plateau between the decreasing variance and the increasing bias. Thus in Figure 2 one takes k between 80 and 110 to obtain the estimate $\hat\lambda=0.28$.

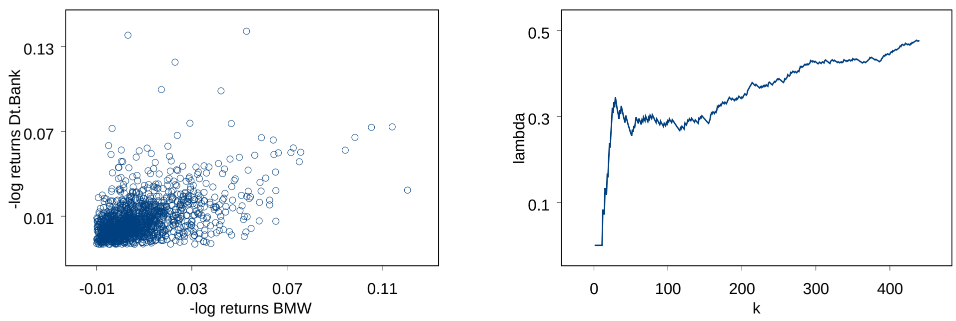

Figure 2. Scatter plot of BMW versus Dt. Bank daily stock log-returns (2325 data points) and the corresponding tail dependence coefficient estimate $\hat{\lambda}$ for various k.

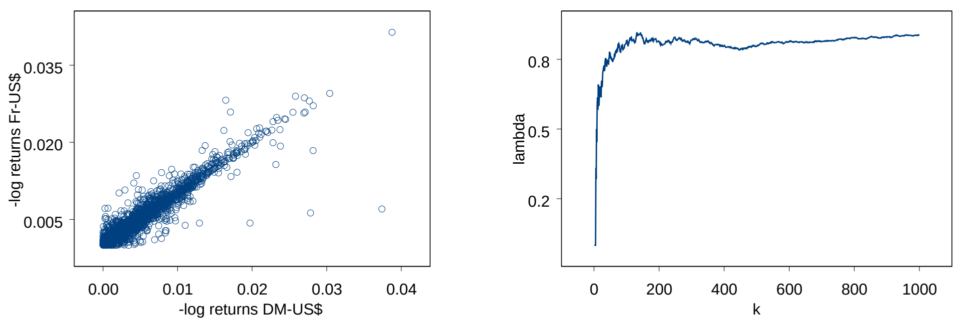

Figure 3. Scatter plot of DM-US$ versus FF-US$ daily exchange rate log-returns (5000 data points) and the corresponding tail dependence coefficient estimate $\hat{\lambda}$ for various k.

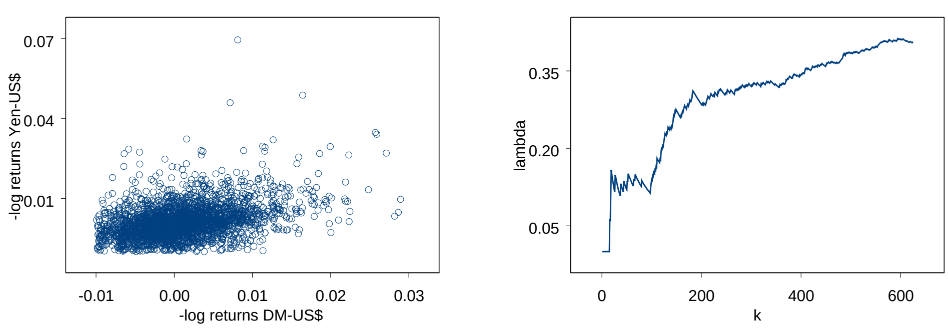

Figure 4. Scatter plot of DM-US$ versus Yen-US$ daily exchange rate log-returns (3126 data points) and the corresponding tail dependence coefficient estimate $\hat{\lambda}$ for various k.

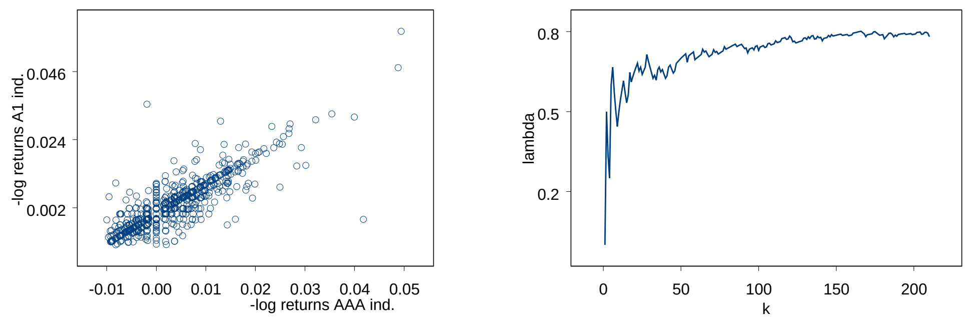

Figure 5. Scatter plot of AAA.Industrial versus A1.Industrial daily log-returns (1043 data points) and the corresponding tail dependence coefficient estimate $\hat{\lambda}$ for various k.

7. Conclusions

Summarizing the results, we have found that elliptical copulae provide appealing dependence structures for asset portfolio modelling within internal credit-risk management. We characterized those elliptical copulae which incorporate dependencies of extremal credit default events by the so-called tail dependence property. Further we showed that most elliptical copulae having the tail dependence property are in the domain of attraction of an extreme value distribution. Thus powerful tools of extreme value theory can be applied. Moreover, the application of elliptical copulae is recommended due to the existence of good estimation and simulation techniques.

Acknowledgment. The author wants to thank Prof. Dr. U. Stadtmüller and Prof. Dr. R. Kiesel for helpful comments and discussions.

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