B. Babel, E. Bomsdorf and R. Schmidt (2006) Forecasting German mortality using panel data procedures, to appear in the *Journal of Population Economics*.

Forecasting German mortality using panel data procedures*

Bernhard Babel, Eckart Bomsdorf, Rafael Schmidt

Department of Economic and Social Statistics, University of Cologne, Albertus-Magnus-Platz, 50923 Cologne, Germany
Email: babel@wiso.uni-koeln.de, bomsdorf@wiso.uni-koeln.de, rafael.schmidt@uni-koeln.de (corresponding author), Fax: +49-221-470-5074

Abstract. Reliable forecasts of life expectancies are of importance for the financial stability of social security systems and the life insurance industry. A discrete-time and a continuous-time stochastic process are proposed to model the dynamics of German mortality rates from which life expectancies are calculated. More precisely, a panel data model is utilized which distinguishes between a common time effect and a common age effect. The model is easy to fit, yields interpretable parameters, and allows for a simple analysis of the forecast error. The main applications of the model are the forecast of mortality rates - and the resulting life expectancies - and the pricing of mortality derivatives.

JEL classification: J11, C51, I12

Key words: Stochastic mortality, panel data model, life expectancy

1 Introduction and motivation

The development of mortality is typically described using mortality rates - also referred to as death rates. The precise assessment of its probabilistic future behavior is important for the financial stability of social security systems, life insurance companies, and related industries.[1] For example, most premium and risk capital calculations of life insurers are based on life tables which summarize the mortality rates of a population. The German Institute of Actuaries (DAV) considers the accurate projection of future death rates as one of the most pressing issues in the German insurance and pension industry. In a recent report (Deutsche Aktuarvereinigung, 2004), the DAV analyzed several deterministic projection models for German mortality rates with different cohort and period trend-functions and came to the conclusion that a log-linear approach is most suitable.[2] Bomsdorf and Trimborn (1992) proposed an extrapolative projection model for future German mortality rates. In their approach, projections of one-year mortality rates $m_x(t)$ depend log-linearly on the individual age x and observation year t. The present paper extends their deterministic model into a time-dynamic stochastic model by utilizing a panel data approach. More precisely, our model makes it possible to distinguish between a common time effect and a common age effect in the time series of cross-sectional mortality data using panel data procedures.

Our model is related to a popular probabilistic mortality model developed by Lee and Carter (1992). Their model is also based on a log-linear approach and has been extended in a number of publications; see Lee (2000) for an overview. It has already been applied to the stochastic modelling and forecasting of mortality rates in various countries, see e.g. Tuljapurkar et al. (2000). The relationship between our mortality model and the Lee-Carter model will be discussed below. It turns out that their model proposed for US mortality rates is not directly applicable to German life tables. In particular, it leads to confidence bands of future life expectancy in Germany which are implausibly narrow.

*All correspondence to Rafael Schmidt. The authors thank two anonymous referees for constructive comments which improved an earlier version of this paper. The third author gratefully acknowledges financial support by the Deutsche Forschungsgemeinschaft (DFG).

Our first objective is to provide forecasts of future German mortality rates with plausible confidence bands - and the resulting life expectancies - and to provide a mortality model with interpretable parameters. Our second objective is to build a stochastic model which serves as a basis for the pricing of financial products depending on mortality fluctuations. Obviously, future cash flows of almost all life insurance products depend on the risk of a changing mortality. Even though the price of this risk is usually not explicitly mentioned in the contract, most insurance companies are aware of it and charge an additional ad-hoc price premium. However, pure mortality products such as the Swiss Re mortality bond issued in December 2003 (which is based on a mortality index of the general population of US, UK, France, Italy and Switzerland), need a precise determination of the mortality risk and the resulting price premium. Furthermore, for internal risk management, insurance companies would like to be able to quantify precisely the risk contribution due to fluctuating mortality. Note that most life insurers hedge themselves against mortality risk by selling products with opposed cash flow structures, such as life and pension insurances.

The paper is organized as follows. After introducing the data in Section 2.1, we describe our model and the estimation of the parameters (Section 2.2). Then, in Section 2.3, we present the results for German life expectancy until 2050. A comparison of our model to Lee and Carter (1992) is given in Section 2.4. The impact of random death rates on the price of mortality derivatives is illustrated in Section 3 where we price a simple mortality bond on the basis of our stochastic model. We conclude in Section 4.

2 Stochastic modelling of mortality

2.1 Data description

The stochastic mortality model that follows will be applied to mortality rates obtained from German period life tables. These life tables are created by the German Federal Statistical Office (Statistisches Bundesamt) over a period of three years and have yearly been published since 1962. In the following we refer to a life table by its middle year, e.g. life table 1961/1963 corresponds to year 1962. The last available life table for our calculations was 2002. The tables contain mortality rates $m_x(t)$ for each gender, individual age x, and observation year t. Age x is given - in one-year steps - from 0 to 89 until the year 2000, and from 0 to 100 since the year 2001. Because of the increasing life expectancy in Germany, we are interested in mortality rates up to the age of 115. For that, we apply the Kannisto model (Thatcher, Kannisto and Vaupel, 1998) in order to extrapolate the last available life table up to the age of 115.[3]

Mortality rates of (former) East Germany increased for a short while after the German reunification in 1990 (see, for example, Riphahn and Zimmermann, 2000), then slowly decreased and approached the level of West German mortality. However, due to the relatively small size of the East German population, this effect had little impact on overall German life expectancy. We disregard this effect and use West German life tables until 2000 which is in line with Bomsdorf (2004). Since 2001, life tables of West and East Germany are not available separately, thus, we took the combined life tables from then on. Note that the considered death rates form a quasi-panel data set.[4]

2.2 The model

The discrete-time model. The projection model by Bomsdorf and Trimborn (1992) for one-year mortality rates $m_x(t)$, depending on age x and year t, takes the form:

$$m_x(t) = \exp(\alpha_x + \beta_x t).$$

Thus, the parameters α_x and β_x are age specific. The gender indexing is suppressed for notational convenience. In order to incorporate the current mortality rate[5] (at time t_0), this log-linear function will be rewritten as:

$$m_x(t) = m_x(t_0) \cdot \exp\{\beta_x(t - t_0)\}. \tag{1}$$

The term $\exp(\beta_x)$ will be interpreted as a growth factor (or reduction factor). More precisely, $100 \cdot \{\exp(\beta_x) - 1\}$ shows the annual percentage change in death rates for x-year old persons: $\beta_x < 0$ indicates a decline of $m_x(t)$, whereas $\beta_x > 0$ implies an increase of $m_x(t)$.

We extend the Bomsdorf-Trimborn model, as given in Formula (1), by expressing the (approximate) *growth rate*[6] β_x via a stochastic time-dependent growth rate $\beta(x,t)$, which accounts for the decline in death rates over time. In other words, $\beta(x,t)$ is modelled as a family of random variables depending on age x and year t, and the dynamics of the mortality rates takes the form

$$m_x(t) = m_x(t-1) \cdot \exp\{\beta(x,t)\}. \tag{2}$$

We utilize a panel data model for $\beta(x,t)$. In particular, we distinguish between two effects: first, a common time effect u_t over all ages (which denotes the common level of mortality growth/decline in year t) and second, an age specific effect μ_x which is stochastically independent of u_t :

$$\beta(x,t) = u_t + \mu_x + \sigma_x \varepsilon_{x,t}. \tag{3}$$

The i.i.d. standardized error terms $\varepsilon_{x,t}$ represent influences which are not captured by the first two summands and σ_x models the nonconstant volatility across age. We model u_t as a stochastic process and μ_x as a fixed effect. Wilmoth (1990) also utilizes an additive approach; however, instead of focusing on the growth rate he models the logarithmic death rate directly. Denton et al. (2005) model $\beta(x,t)$ as a stochastic process, too, and apply it to Canadian mortality data. They propose a 'quasi' version of a vector autoregressive process for $\beta(x,t)$.

Estimation. The fitting procedure is based on known techniques for panel data analysis, see e.g. Baltagi (2005). In order to obtain a unique parametrization of the model, we set the median of μ_x over all ages x equal to zero. First, we estimate u_t as the empirical median of $\beta(x,t)$ taken over all ages x and denote the estimator by $\hat{u}_t$. Contrary to the sample mean, the median is quite robust regarding the varying volatility σ_x and the resulting outliers of $\beta(x,t)$ (see Appendix for details). The fixed effect μ_x is now estimated by $\hat{\mu}_x = \bar{\beta}(x,\cdot) - \bar{u}_{\cdot}$. In the second step, we estimate the parameters of the stochastic process u_t which, in the discrete-time setting, is given by

$$u_{t+1} = u_t + r \cdot (s - u_t) + \eta_t. \tag{4}$$

Here, η_t denotes the i.i.d. normally distributed error terms with $E(\eta_t) = 0$ and $V(\eta_t) = \sigma_\eta^2$. The parameter s is consistently estimated by $\bar{u}_{\cdot}$. Further, the parameter r is obtained using the least square estimator

$$\hat{r} = \arg\min_r \sum_{t=1}^{T-1} \left\{\hat{u}_{t+1} - \hat{u}_t - r \cdot (\hat{s} - \hat{u}_t)\right\}^2 .$$

Finally, the estimators for the variances of the error terms are

$$\hat{\sigma}_\eta^2 = \frac{1}{T-2} \sum_{t=1}^{T-1} \left\{\hat{u}_{t+1} - \hat{u}_t - \hat{r} \cdot (\hat{s} - \hat{u}_t)\right\}^2 \quad \text{and} \quad \hat{\sigma}_x^2 = \frac{1}{T-1} \sum_{t=1}^{T} \left\{\beta(x,t) - \bar{\beta}(x,.) - \bar{u}. - \hat{u}_t\right\}^2 .$$

In order to get an impression of the magnitude of μ_x and σ_x, we present the estimation results for men in Figure 1. The results for women are similar. They clearly show that the different volatilities σ_x across all ages cannot be ignored in the mortality forecast.

Further, our estimation yields an empirical mean of -0,021 for u_t for women and -0,018 for men. This indicates an average yearly mortality decline of 2,1% for women and 1,8% for men. More estimation results are presented in the next section.

The continuous-time model. The plot in Figure 2 displays the realized path of the process u_t based on the life tables from 1962 to 2002 and one example path of u_t according to our fitted model until 2050 for women.[7] It also shows that - in the past - the process $(u_t)_{t\geq 0}$ tended back to its mean. In particular, we identified that a mean reverting autoregressive process with normally distributed error terms fits u_t well. This mean reversion motivates the following continuous-time embedding using an Ornstein-Uhlenbeck process - in finance known as a Vasicek model - which is given by

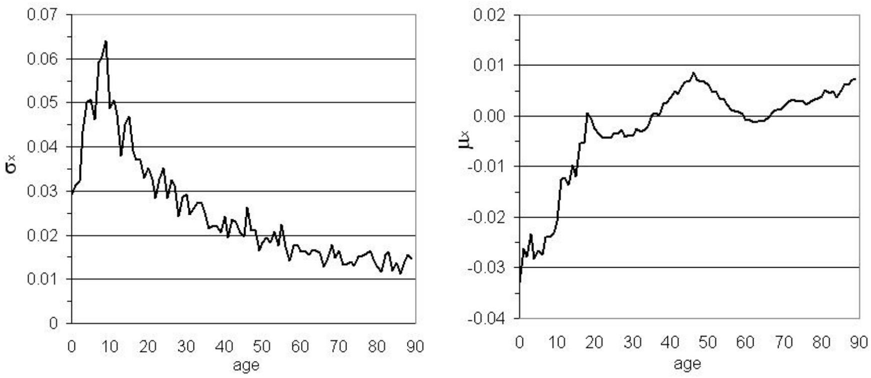

Figure 1: Estimates of the volatility σ_x across all ages (left plot) and estimates of the fixed age effects μ_x (right plot) across all ages for men.

$$du_t = r\,(s - u_t)\,dt + \sigma dW_t,\ \sigma > 0. \tag{5}$$

Note that in the continuous-time setting, Formula (2) is replaced by a stochastic differential equation (SDE). W_t is a Wiener process and the parameter $r \in [0,1]$ expresses the degree of the backwards drift to the mean s. If r is close to one, the degree of mean reversion is high. In the discrete setting, the special case $r = 1$ yields a white noise process and $r = 0$ represents a random walk. In contrast to life-table analysis, where the mortality dynamics are sufficiently described via discrete-time models, many pricing models in finance are easier to deal with in a continuous-time framework. This is mainly due to various results regarding the risk-neutral evaluation of financial products, see Bingham and Kiesel (2004) for more details.

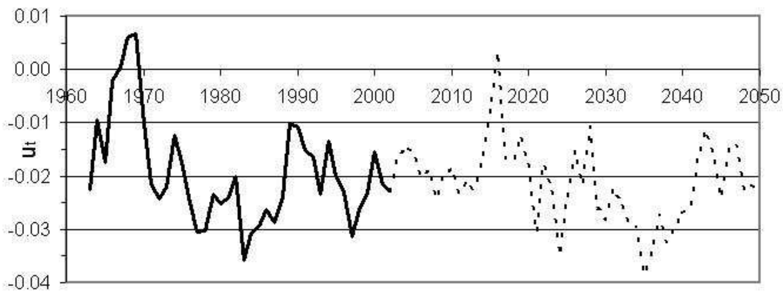

Figure 2: Realized path of u_t from 1963 to 2002 (solid line) and example path (dotted line) according to our fitted model for women.

2.3 Forecasting German life expectancy

As mentioned in Section 2.1, we apply the Kannisto model (Thatcher, Kannisto and Vaupel, 1998) in order to extrapolate the last available life table of 2002 up to the age of 115. In addition, we assume that the average annual mortality decline at the age of $x = 115$ equals zero, i.e., $E(u_t + \mu_{115}) = 0$. Between the ages of 90 and 115, we interpolate μ_x linearly. Further, we set $\sigma_x = \sigma_{89}$ for all $x = 90, \ldots, 115$. In order to illustrate the benefits of our model, we consider four variants of u_t (the parametrization is that of Formula (5)): first, a deterministic variant (A), second, a white noise process with $r = 1$ (B.1), third, a fitted Ornstein-Uhlenbeck variant with $r = 0.28$ for women and $r = 0.34$ for men (B.2) and, finally, a random walk with $r = 0$ (B.3).

Figure 3 illustrates the mortality levels $m_x(2002)$, and the forecasted medians and approximate 90% forecast intervals of $m_x(2050)$ for the female population on a logarithmic scale.[8] It implies a strong decline of $m_x(t)$ from $t = 2002$ to $t = 2050$ across all ages. Even though the biggest

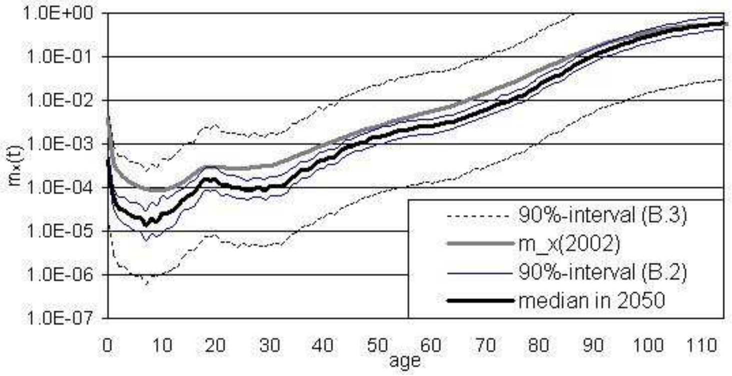

Figure 3: Mortality rates $m_x(t)$ (on logarithmic scale) in 2002 (gray line), forecasted median (thick black line) and 90% forecast intervals[7] in 2050 for the female population using variant B.2 (thin solid line) and B.3 (dotted line).

changes (on logarithmic scale) are found in the younger age groups, the decrease for older age groups affects the life expectancy and population development more due to much higher death rates. The approximate 90% forecast interval for the fitted Ornstein-Uhlenbeck variant is much narrower than the 90% forecast interval for the random walk. For the latter, the boundaries of the 90% interval for the age 100 reach from a logarithmic death rate of 1.00 to a rate of about 0.01. Thus, the random walk sharply overestimates the variance and - in our context - obviously does not represent a plausible model for u_t.

Table 1 contains the results for the life expectancy of men (m) and women (f) at birth in 2050, based on forecasted period life tables. In line with a number of other authors, e.g. Lee and Carter (1992), we do not forecast life expectancies directly but calculate those from forecasted mortality rates.

	Type	5% quantile women (men)	median women (men)	95% quantile women (men)	band width women (men)
Variant A		88.0 (82.1)	88.3 (82.4)	88.5 (82.7)	0.4 (0.6)
Variant B.1	I	87.4 (81.3)	88.3 (82.4)	89.2 (83.6)	1.9 (2.4)
	II	86.9 (80.4)	88.3 (82.4)	89.8 (84.4)	2.9 (4.0)
Variant B.2	I	86.1 (79.5)	88.3 (82.4)	90.5 (85.3)	4.4 (5.8)
	II	84.8 (77.5)	88.3 (82.4)	92.6 (86.9)	7.8 (9.4)

Table 1: Life expectancy at birth in 2050 based on the period model - in years - for women (and men). Type II includes the uncertainty in the estimation of the model parameters in Formulae (3) and (4), whereas Type I excludes the latter; see the Appendix for more details.

Our results demonstrate an increase in life expectancy for both genders. In the period model, median life expectancy at birth for women (respectively for men) rises from 81.3 (75.6) years 2002 to 88.3 (82.4) years in 2050.[9] Variant A with a deterministic u_t yields a band width of only 0.4 years for women (0.6 years for men). The white noise variant B.1 generates a band width of 1.9 (2.4) years, and 2.9 (4.0) if we include the uncertainty in the parameter estimation. The fitted Ornstein-Uhlenbeck variant B.2 produces plausible band widths of 4.4 (5.8) years without the uncertainty in the parameter estimation and 7.8 (9.4) including the latter. In all cases, the forecast intervals for men are wider than for women. Thus, future male mortality is more volatile than female mortality.

In contrast to the period view, Table 2 provides life expectancies based on the cohort model.[10] In the cohort model, median life expectancy at birth for women (respectively for men) increases from 90.8 (84.8) years 2002 to 96.2 (90.1) years in 2050. The proposed model B.2 leads to a 5% quantile of 92.7 (85.9) years for women (for men) and a 95% quantile of 99.6 (94.0) years. If we include the uncertainty from the parameter estimation, the 5% quantile becomes 89.4 (81.0) and the 95% quantile is 104.1 (101.3) years.

	Type	5% quantile women (men)	median women (men)	95% quantile women (men)	band width women (men)
Variant A		95.8 (89.6)	96.2 (90.1)	96.6 (90.5)	0.8 (0.9)
Variant B.1	I	94.9 (88.5)	96.2 (90.1)	97.4 (91.9)	2.5 (3.4)
	II	93.2 (85.6)	96.2 (90.1)	99.5 (94.9)	6.3 (9.3)
Variant B.2	I	92.7 (85.9)	96.2 (90.1)	99.6 (94.0)	6.9 (8.1)
	II	89.4 (81.0)	96.2 (90.1)	104.1 (101.3)	14.7 (20.3)

Table 2: Life expectancy at birth in 2050 based on the cohort model - in years - for women (and men). Type II includes the uncertainty in the estimation of the model parameters in Formulae (3) and (4), whereas Type I excludes the latter; see the Appendix for more details.

Tables 1 and 2 show that the forecasted life expectancies are quite different for the period model and the cohort model. We emphasize that the median life expectancies in the cohort model are higher than in the period model, although the absolute increase is smaller in the cohort model. In particular, the absolute increase of the median life expectancy in 2050 for women (for men) is 7.0 (6.8) years in the period model whereas it is 5.4 (5.3) in the cohort model. The confidence intervals are wider in the cohort model since we need to forecast mortality rates further ahead.

A comparison of the above results for the different processes u_t shows that versions A and B.1 systematically underestimate the volatility and produce forecast bands which are implausibly narrow. By contrast, version B.3 yields extremely wide forecast bands. In other words, these versions of u_t are not adequate in order to model German mortality data (see also Figure 2). Version B.2, which models the process u_t as a mean-reverting autoregressive process in discrete time or as an Ornstein-Uhlenbeck process in continuous time, yields much more plausible confidence intervals for the forecast of life expectancy and death rates.

Finally, we compared the previous findings with life expectancies derived from projected death rates using the deterministic Bomsdorf-Trimborn model given in Section 2.2. The calculated life expectancies deviate at most by 0.1 years from the median life expectancies given in Tables 1 and 2 (in the column 'median'). Further results for future life expectancies in Europe, North America, Japan, and Australia - derived from the above stochastic mortality model - are reported in Babel et al. (2006).

2.4 Comparison to Lee and Carter (1992)

The log-linear stochastic mortality model developed by Lee and Carter (1992) takes the following form:

$$m_x(t) = \exp(a_x + b_x k_t + \varepsilon_{x,t}), \tag{6}$$

where the exponential trend function contains age-dependent parameters a_x and b_x, and k_t is modelled as a general time-varying index.[11] The random variables $\varepsilon_{x,t}$ reflect stochastic influences which are not captured by the previous terms. In a large number of papers this model has been extended and utilized for the stochastic modelling and forecasting of mortality rates in various countries, see e.g. Lee (2000) for an overview and Li et al. (2004) for using the model for populations with limited data.

Lee and Carter (1992) propose a fitting procedure which is based on a least square method - implemented via a singular value decomposition - regarding the matrix of logarithms of $m_x(t)$. Comparing the Lee-Carter model to our model, a first difference is that we focus on the logarithmic difference of $m_x(t)$ rather than on the logarithm of $m_x(t)$ itself. In this context, if the error term is ignored, Formula (6) yields

$$\ln m_x(t) - \ln m_x(t-1) = b_x(k_t - k_{t-1}). \tag{7}$$

Thus, the growth factor $\beta(x,t)$ has a multiplicative structure rather than an additive structure, as we proposed in Formula (3). For US mortality data, Lee and Carter (1992) identify a random walk with drift for k_t, i.e. $k_t = c + k_{t-1} + e_t$. In that case, Formula (7) takes the form

$$\ln m_x(t) - \ln m_x(t-1) = b_x c + b_x e_t = (b_x - 1)c + c + b_x e_t.$$

It follows that the time-varying factor u_t in Formula (3) is modelled as a constant c in the above model. This highlights a second difference to our model where we propose an Ornstein-Uhlenbeck process for u_t. The additive structure of our stochastic mortality model has further implications: First, we are able to incorporate the varying volatility of death rates across the different age groups. In particular, we decompose the volatility into a purely temporal term and a statistically independent age-specific term. Second, the interpretation of the parameters of our model are that of well-established panel data models. The decomposition into the age effect and time effect allows a direct and intuitive interpretation of the yearly mortality fluctuation. Third, we do not need to estimate the parameters using a singular value decomposition - as in the Lee-Carter model - which may require an additional re-estimation of the time index k_t. We refer to Lee (2000) for a discussion of the latter.

In order to measure the goodness of fit of the respective mortality models, we consider two different approaches. First, we assess the goodness of fit of the projected (future) death rates by a measure which has also been utilized in Lee and Carter (1992). Second, we test whether the hypothesis of white noise for u_t must be rejected and examine its implications on the width of the confidence bands of future life expectancies. Regarding the first approach, the fit of our model (version B.2) in comparison to the Lee-Carter model is demonstrated in Table 3. This table presents - for various age groups - one minus the ratios of the sample variance of the differences between the actual and projected death rates to the sample variance of the actual rates. A projected death rate in our model in a given year is based on the (realized) death rate in the preceding year multiplied by the average growth factor which is estimated as in Section 2.2, cf. also Formula (2). The Lee-Carter model is estimated as described in Lee and Carter (1992): We estimate the parameters using the singular value decomposition and normalize the b_x to sum to unity and the k_t to sum to 0. We do not re-estimate the k_t in order to improve the fit to the actual observed deaths, as this would decrease the goodness of fit with respect to the above measure.

	Babel-Bomsdorf-Schmidt		Lee-Carter	
age group	male (in %)	female (in %)	male (in %)	female (in %)
0-19	98.8	97.9	96.2	98.7
20-39	97.9	98.3	95.8	98.9
40-59	98.0	98.7	92.9	98.7
60-79	99.0	99.5	96.2	99.5
80-89	98.6	99.1	94.4	99.0
min	96.5	95.3	84.3	96.0
max	99.5	99.6	99.5	99.8
average	98.6	98.7	95.2	98.9

Table 3: Average explained variance of our mortality model (version B.2) in comparison to the Lee-Carter model.

The results in Table 3 reveal that the fit of the projected death rates in both models is satisfactory. Our model leads to a better fit for men whereas Lee-Carter exceeds our model for women. In particular, the lowest explained variance in our model (in the Lee-Carter model) is 0.953 for women aged 21 (respectively 0.843 for men aged 18); the average explained variance is 0.986 (0.952) for men and 0.987 (0.989) for women.

In our second approach, we first test whether the hypothesis of white noise for u_t must be rejected. The Ljung-Box test clearly rejects the above null-hypothesis at a significance level of 0.1% for both women and men, cf. also Figure 2. The impact of substituting the white noise process by a mean reverting process for u_t on the width of the confidence bands for future life expectancies is shown in Table 4.

As expected, the width of the confidence bands in the Lee-Carter model are very similar to the results for the model version B.1, which assumes a white noise process for u_t. In contrast to that,

	Type	5% quantile women (men)	median women (men)	95% quantile women (men)	band width women (men)
Lee-Carter model	I	87.5 (81.6)	88.4 (82.7)	89.2 (83.8)	1.7 (2.2)
	II	87.1 (81.1)	88.4 (82.7)	89.7 (84.4)	2.6 (3.3)
Variant B.1	I	87.4 (81.3)	88.3 (82.4)	89.2 (83.6)	1.9 (2.4)
	II	86.9 (80.4)	88.3 (82.4)	89.8 (84.4)	2.9 (4.0)
Variant B.2	I	86.1 (79.5)	88.3 (82.4)	90.5 (85.3)	4.4 (5.8)
	II	84.8 (77.5)	88.3 (82.4)	92.6 (86.9)	7.8 (9.4)

Table 4: Life expectancy at birth in 2050 based on the period model - in years - for women (and men). Type II includes the uncertainty in the estimation of the model parameters in Formulae (3) and (4), whereas Type I excludes the latter; see the Appendix for more details.

the confidence bands of our model version B.2 - where u_t is modelled as a mean reverting process - are more than twice the band width calculated from model version B.1.

Note that the confidence bands obtained by Tuljapurkar et al. (2000) for German life expectancies are not directly comparable to our results, because the authors use abridged mortality data for their estimation and consider a different observation period (1952-1990). Regarding the first, we prefer the period life tables, as described in Section 2.1, for our calculation because these tables are commonly used in the German insurance and pension industry.

3 Pricing mortality bonds

Our second application refers to the pricing of financial products which depend on mortality fluctuations. We have already mentioned that future cash flows of almost all life insurance products depend on the risk of a changing mortality. In the following we consider a simple mortality bond with maturity T and the following coupon payoff-structure (a similar structure has been considered by Lin and Cox, 2005; for other mortality derivatives, see Cairns et al., 2006):

$$C_t = \begin{cases} 0 & \text{if} \quad l_{x+t} > U_t, \\ P \cdot \dfrac{U_t - l_{x+t}}{U_t - L_t} & \text{if} \quad L_t < l_{x+t} \leq U_t, \\ P & \text{if} \quad l_{x+t} \leq L_t, \end{cases} \qquad \text{for } t \leq T.$$

Here, C_t denotes the total coupon paid to the investor in year t and l_{x+t} is the number of survivors from a group of annuitants (in the reference portfolio) initially all aged x. The investor receives the full payment P if l_{x+t} does not exceed a time-dependent lower trigger level L_t, otherwise he gets only a fraction of P. In particular, the investor receives no payments in year t if the number of survivors exceeds the upper level U_t.

This payoff structure comprises the following features: It hedges the risk that the mortality exceeds a pre-specified level. This risk is called *longevity risk*. For example, the protection seller might hedge his risk that the annuity or pension payments exceed a certain level. From the protection buyer's view, this type of mortality derivative is interesting, for example, for pure life insurers or reinsurers who want to diversify their risk of declining mortality. In particular, the payoff structure is that of a so-called double barrier option which is commonly traded with a stock as underlying asset. Another important characteristic of the above payoff structure is its lack of basis risk which comprises the risk that the hedge is not the same as the protection seller's risk. This is often the case if the payment structure depends on a non-traded index.

For our calculations, we assume an initial cohort of 10,000 men or women, all aged 65 or 45 in 2002. The strike levels L_t are determined by the expected number of survivors at time t and we set $U_t = L_t + 500$. The principal amount of the bond (paid at maturity T) is 100 and payment P corresponds to a coupon rate of 5%. Consequently, the price of a straight bond (without mortality risk) discounted at a rate of 5% is 100. The mortality bond is now either evaluated under the risk neutral market measure or the real world measure, depending on whether the instrument (or its underlying asset) is liquidly traded on the market or whether it is evaluated in a risk management context. Note that mortality bonds cannot be perfectly hedged, as it is often the case for equity derivatives. Here we price the mortality bond using the real-world death rates based on the cohort model (by taking the corresponding expectation of the payoffs). Some results are presented in Table 5.

	male (65)	male (45)	female (65)	female (45)
Number of annuitants	10,000	10,000	10,000	10,000
Straight bond price	100.00	100.00	100.00	100.00
Mortality bond price A	98.37	99.58	98.90	99.78
Mortality bond price B.1	97.04	99.13	97.90	99.61
Mortality bond price B.2	93.12	97.99	95.34	99.15
Coupon rate	0.05	0.05	0.05	0.05
Maturity (in years)	26	26	26	26

Table 5: Mortality bond prices for variants of the mortality process based on the cohort model.

Table 5 illustrates a declining bond price with increasing parameter r in the Ornstein-Uhlenbeck process (5). This decline is intuitive as the coupon payments are more likely to be lower in case the fluctuations of the death rates increase. Further, it shows that the evaluation of this type of mortality derivative is quite sensitive to the stochastic modelling of mortality.

4 Conclusion

We propose a stochastic mortality model which forms an extension of the deterministic log-linear projection model by Bomsdorf and Trimborn (1992). The latter describes the future development of mortality rates, starting from the current mortality level, by age dependent growth factors. We extend the model into a time-dynamic stochastic model by utilizing a panel data approach. More precisely we distinguish between two effects: first, a common time effect over all ages which denotes the common level of mortality growth/decline, implemented by a mean reverting process, and second, an age specific effect, which describes the age specific deviation from the common mortality level. The structure of the model allows a direct and intuitive interpretation of the parameters and leads to plausible forecasting results, e.g. in Germany, life expectancy at birth for women (for men) rises from 81.3 (75.6) in years 2002 to 88.3 (82.4) years in 2050 in the period model. Further, the period model implies a 95% quantile of 92.6 (86.9) years for women (for men) in 2050. Finally, the forecasted mortality rates are utilized for the pricing of a simple mortality bond whose payoff structure depends on the changing mortality.

The confidence bands for life expectancy are wider in our model than in the Lee and Carter (1992) model applied to German life tables. This is due to an additional temporal noise term which is found in German life tables. By contrast, the goodness of fit of the (deterministic) projection of future death rates is quite similar in both models.

Summarizing the characteristics of our stochastic mortality model: The volatility is decomposed into a purely temporal term and a stochastically independent age-specific term. This decomposition leads to an intuitive interpretation of the model and its parameters. Further, we incorporate the varying volatility of death rates across the different age groups. The estimation procedure is that of well-established panel data procedures.

Appendix: Forecast error

The error structure of the forecast of the growth rate $\beta(x,t)$ is that of a panel data or error component model with heteroskedasticity and serial correlation, cf. Chapter 5 in Baltagi (2005). Let $e_{x,t}$ denote the error of the forecast for $\beta(x,t)$ given the information up to time $t-1$. Then, $e_{x,t} = \varepsilon_{u_t} + \varepsilon_{\mu_x} +$

$(\sigma_x + \varepsilon_{\sigma_x})\varepsilon_{x,t}$, where ε_{u_t} is the error in forecasting u_t, and ε_{μ_x} (resp. ε_{σ_x}) is the error in estimating the parameter μ_x (resp. σ_x). The empirical bootstrap shows that the standard deviation of ε_{μ_x} (which is on average 0.003) is negligible with respect to the standard deviation σ_x and may, hence, be disregarded in the derivation of the forecast interval. This holds also for the respective correlations between ε_{u_t}, ε_{σ_x} and ε_{μ_x} over all ages. The distribution of the error ε_{u_t} in our model version B.2 is determined by the error η_t and the error in estimating the parameters r, s, and σ_η given in Formula (4). The empirical standard deviation of the estimation of r, s, and the estimated standard deviation σ_η of η_t are provided in Table 6.

	$\hat{\sigma}_r$	$\hat{\sigma}_s$	$\hat{\sigma}_\eta$
male	0.1221	0.0043	0.0088
female	0.1147	0.0040	0.0071

Table 6: Standard error of the estimation of r, s, and the standard deviation σ_η.

The forecast intervals of Type I in Tables 1, 2, and 4 are based on the error terms ε_{u_t} and $\sigma_x\varepsilon_{x,t}$ excluding the error in the parameter estimation, and are derived from 5000 Monte Carlo simulations of successive one-year growth rates (for each age). The Type II forecasts in these tables include the errors in the estimation of the model parameters σ_x, r, s, and σ_η; the distribution of these errors is assumed to be multivariate normal. Obviously, with increasing forecast horizon, the error of the time effect ε_{u_t} dominates the error term $\sigma_x\varepsilon_{x,t}$.

Table 7 illustrates the sensitivity of the estimation procedure by fitting the model either from all available death rates or from death rates for age groups 30-89 or 60-89 only. It implies that the model forecasts and estimates are quite robust if derived from those death rates. Finally, our calculations show that if we disregard a small (random) number of historical life tables the estimation results are stable.

age group	0-89	30-89	60-89
5%-quantile	19.2	19.3	19.2
95%-quantile	24.5	24.6	24.4
$\hat{r}$	0.34	0.32	0.31
$\hat{\sigma}_\eta \cdot 100$	0.88	0.85	0.86

Table 7: Estimation results of parameters r and σ_η, and life expectancy in 2050 for men aged 60 (cohort model) based on death rates of age groups 0-89, 30-89, and 60-89

Notes

[1]Moreover, Yakita (2001) shows the effects of changing mortality on fertility, capital accumulation, and economic growth.

[2]See also, Denuit and Goderniaux (2005) who demonstrate that log-linear models provide accurate projections of Belgian mortality rates.

[3]Thatcher, Kannisto, and Vaupel (1998), p.30, and the German Institute of Actuaries (Deutsche Aktuarvereinigung, 2004, pp.79–83) analyze the goodness of fit of various extrapolation methods for ages beyond $x = 100$. Both references conclude that the Kannisto model fits the empirical data well. Analogously to Thatcher, Kannisto, and Vaupel (1998) we base the estimation of the parameters in the Kannisto model on the ages $x = 80$ to 98.

[4]We mention that further 11 life tables of Germany are available which are directly based on censuses, the first one from 1876 and the last one from 1987. In contrast to those life tables, the above series of life tables is derived on the basis of observed mortality data - without smoothing - and on extrapolated population data which are based on those censuses.

[5]The insertion of the most recently available mortality rate, as initial value, into the mortality projection model is common practice in life insurance. This approach guarantees that the short-term forecasts have a smooth transition from the most recent mortality rate.

[6]Note that for β_x close to zero we have $\beta_x \approx e^{\beta_x} - 1$.

[7]The realized paths of u_t for men and women - except for one data point in 2000 - are nearly parallel. Motivated by this, we also applied a bivariate stochastic process (u_t) for men and women regarding the estimation of life expectancy. In simultaneous calculations such as population projections it is important to consider this kind of dependence. In contrast, this dependence does not influence our separate forecasts of life expectancy.

[8]We choose the median as a robust estimator of the mean. In comparison to variant B.2, the forecast bands for variant B.1 are closer to the median. The corresponding results for men are similar. The confidence bands in Figure 3 do not include the uncertainty in the parameter estimation.

[9]These median life expectancies at birth in 2050 for the period model are similar to the results obtained by Bomsdorf (2004) and closely correspond to the upper variant (out of three) of life expectancy projections by the German Federal Statistical Office (Statistisches Bundesamt, 2003). Oeppen and Vaupel (2002) even expect a life expectancy above 100 years for women at birth in 2050 which is included in the 90% confidence band of our cohort model.

[10] For the cohort model and the resulting life expectancies, the following projected period mortality rates are considered $m_0(t), m_1(t+1), \ldots, m_{115}(t+115)$. For projections of German future mortality rates based on a deterministic cohort model we refer to Bomsdorf (2002).

[11]The special case of $k_t = t$ is closely related to the deterministic approach by Bomsdorf and Trimborn. Denuit and Goderniaux (2005) also model $k_t = t$ for Belgian mortality data. An extensive review of the literature on mortality models is given in Pitacco (2004).

References

Babel B, Bomsdorf E, Schmidt R (2006) Future life expectancy in Europe, North America, Japan and Australia. *Submitted*. Retrievable from http://www.uni-koeln.de/wiso-fak/wisostatsem/autoren/schmidt.

Baltagi BH (2005) *Econometric Analysis of Panel Data.* Third Edition, John Wiley & Sons, Chichester.

Bingham NH, Kiesel R (2004) *Risk neutral evaluation: Pricing and Hedging of Financial Derivatives.* Springer Finance Series, New York.

Bomsdorf E (2002) *Neue Generationensterbetafeln für die Geburtsjahrgänge 1933-2003.* Josef Eul Verlag, Köln.

Bomsdorf E (2004) Life expectancy in Germany until 2050. *Experimental Gerontology* 39:159–163.

Bomsdorf E, Trimborn M (1992) Sterbetafel 2000. Modellrechnungen der Sterbetafel. *Zeitschrift für die gesamte Versicherungswissenschaft* 81:457–485.

Cairns AJG, Blake D, Dowd K (2006) Pricing Death: Frameworks for the Valuation and Securitization of Mortality Risk. *Astin Bulletin* 36:79–120.

Denton FT, Feaver CH, Spencer BG (2005) Time series analysis and stochastic forecasting: An econometric study of mortality and life expectancy. *Journal of Population Economics* 18(2):203–227.

Denuit M, Goderniaux AC (2005) Closing and Projecting Life Tables using Log-Linear Models. *Mitteilungen der Schweizerischen Aktuarvereingung* 1:29–49.

Deutsche Aktuarvereinigung (2004) *Herleitung der DAV-Sterbetafel 2004R für Rentenversicherungen.* Internal Report.

Lee RD (2000) The Lee-Carter method for forecasting mortality, with various extensions and applications. *North American Actuarial Journal* 4:80–93.

Lee RD, Carter LR (1992) Modeling and forecasting US mortality. *Journal of the American Statistical Association* 87:659–671.

Li N, Lee RD, Tuljapurkar S (2004) Using the Lee-Carter Method to Forecast Mortality for Populations with Limited Data. *International Statistical Review* 72:19–36.

Lin Y, Cox SM (2005) Securitization of mortality risks in life annuities. *Journal of Risk and Insurance* 72:227–252.

Oeppen J ,Vaupel JW (2002) Broken limits to life expectancy. *Science* 296:1029–1031.

Pitacco E (2004) Survival models in a dynamic context: a survey. *Insurance: Mathematics and Economics* 35:279–298.

Riphahn RT, Zimmermann KF (2000) *The Mortality Crisis in East Germany.* In: Cornia GA, Paniccià R (eds) The Mortality Crisis in Transitional Economies. Oxford University Press, 227–252.

Statistisches Bundesamt (2003) *Bevölkerung Deutschlands bis 2050.* Wiesbaden. Retrievable from http://www.destatis.de/presse/deutsch/pk/2003/Bevoelkerung_2050.pdf

Thatcher AR, Kannisto V, Vaupel JW (1998) *The force of mortality at ages 80–120.* Monographs on Population Aging 5, Odense University Press, Viborg, Denmark.

Tuljapurkar S, Li N, Boe C (2000) A universal pattern of mortality decline in the G7 countries. *Nature* 405:789–792.

Wilmoth JR (1990) Variation in vital rates by age, period, and cohort. *Sociological Methodology* 20:295–335.

Yakita A (2001) Uncertain lifetime, fertility and social security. *Journal of Population Economics* 14:635–640.