1 Tail dependence

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1.1 Introduction

Tail dependence describes the amount of dependence in the tail of a bivariate distribution. In other words, tail dependence refers to the degree of dependence in the corner of the lower-left quadrant or upper-right quadrant of a bivariate distribution. Recently, the concept of tail dependence has been discussed in financial applications related to market or credit risk, Hauksson et al. (2001) and Embrechts et al. (2003). In particular, tail-dependent distributions are of interest in the context of Value at Risk (VaR) estimation for asset portfolios, since these distributions can model dependence of large loss events (default events) between different assets.

It is obvious that the portfolio's VaR is determined by the risk behavior of each single asset in the portfolio. On the other hand, the general dependence structure, and especially the dependence structure of extreme events, strongly influences the VaR calculation. However, it is not known to most people which are not familiar with extreme value theory, how to measure and model dependence, for example, of large loss events. In other words, the correlation coefficient, which is the most common dependence measure in financial applications, is often insufficient to describe and estimate the dependence structure of large loss events, and therefore frequently leads to inaccurate VaR estimations, Embrechts et al. (1999). The main aim of this chapter is to introduce and to discuss the so-called tail-dependence coefficient as a simple measure of dependence of large loss events.

Kiesel and Kleinow (2002) show empirically that a precise VaR estimation for asset portfolios depends heavily on the proper specification of the tail-dependence structure of the underlying asset-return vector. In their setting, different choices of the portfolio's dependence structure, which is modelled by a copula function, determine the degree of dependence of large loss events. Motivated by their empirical observations, this chapter defines and explores the concept of tail dependence in more detail. First, we define and calculate tail dependence for several classes of distributions and copulae. In our context, tail dependence is characterized by the so-called tail-dependence coefficient (TDC) and is embedded into the general framework of copulae. Second, a parametric and two nonparametric estimators for the TDC are discussed. Finally, we investigate some empirical properties of the implemented TDC estimators and examine an empirical study to show one application of the concept of tail dependence for VaR estimation.

1.2 What is tail dependence?

Definitions of tail dependence for multivariate random vectors are mostly related to their bivariate marginal distribution functions. Loosely speaking, tail dependence describes the limiting proportion that one margin exceeds a certain threshold given that the other margin has already exceeded that threshold. The following approach, as provided in the monograph of Joe (1997), represents one of many possible definitions of tail dependence.

Let $X = (X_1, X_2)^\top$ be a two-dimensional random vector. We say that X is (bivariate) *upper tail-dependent* if:

$$\lambda_U \stackrel{\text{def}}{=} \lim_{v \to 1^-} \mathrm{P}\left\{X_1 > F_1^{-1}(v) \mid X_2 > F_2^{-1}(v)\right\} > 0, \tag{1.1}$$

in case the limit exists. F_1^{-1} and F_2^{-1} denote the generalized inverse distribution functions of X_1 and X_2, respectively. Consequently, we say $X = (X_1, X_2)^\top$ is *upper tail-independent* if λ_U equals 0. Further, we call λ_U the upper tail-dependence coefficient (upper TDC). Similarly, we define the lower tail-dependence coefficient, if it exists, by:

$$\lambda_L \stackrel{\text{def}}{=} \lim_{v \to 0^+} \mathrm{P}\left\{X_1 \leq F_1^{-1}(v) \mid X_2 \leq F_2^{-1}(v)\right\}. \tag{1.2}$$

A generalization of bivariate tail dependence, as defined above, to multivariate tail dependence can be found in Schmidt and Stadtmüller (2003).

In case $X = (X_1, X_2)^\top$ is standard normally or t-distributed, formula (1.1) simplifies to:

$$\lambda_U = \lim_{v \to 1^-} \lambda_U(v) \overset{\text{def}}{=} \lim_{v \to 1^-} 2 \cdot \mathrm{P}\left\{X_1 > F_1^{-1}(v) \mid X_2 = F_2^{-1}(v)\right\}. \tag{1.3}$$

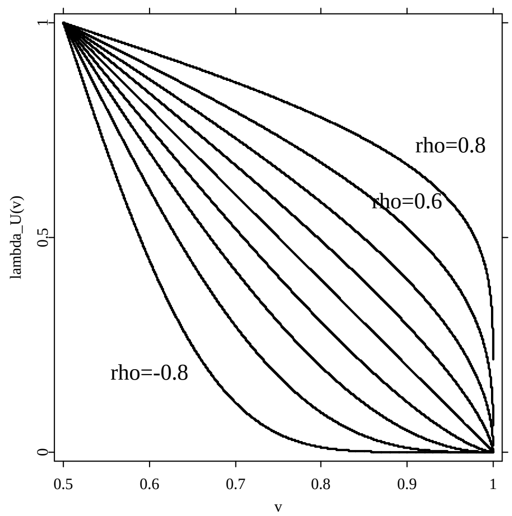

Figure 1.1: The function $\lambda_U(v) = 2 \cdot \mathrm{P}\{X_1 > F_1^{-1}(v) \mid X_2 = F_2^{-1}(v)\}$ for a bivariate normal distribution with correlation coefficients $\rho = -0.8,\ -0.6, \ldots, 0.6,\ 0.8$. Note that $\lambda_U = 0$ for all $\rho \in (-1, 1)$.

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Figures 1.1 and 1.2 illustrate tail dependence for a bivariate normal and t-distribution. Irrespectively of the correlation coefficient ρ, the bivariate normal distribution is (upper) tail independent. In contrast, the bivariate t-distribution exhibits (upper) tail dependence and the degree of tail dependence is affected by the correlation coefficient ρ.

The concept of tail dependence can be embedded within the copula theory. An n-dimensional distribution function $C : [0,1]^n \to [0,1]$ is called a copula if it has one-dimensional margins which are uniformly distributed on the interval $[0,1]$. Copulae are functions that join or "couple" an n-dimensional distribution

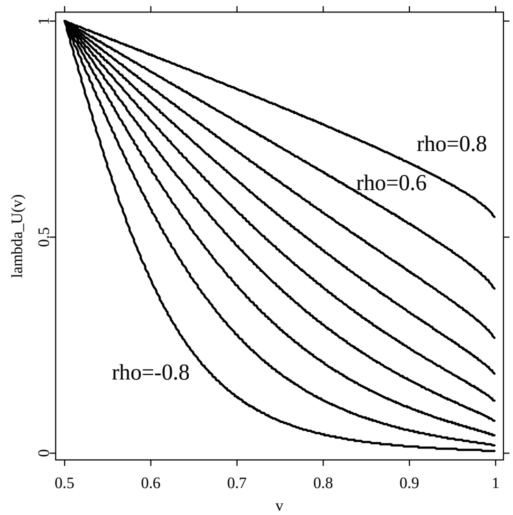

Figure 1.2: The function $\lambda_U(v) = 2 \cdot \mathrm{P}\{X_1 > F_1^{-1}(v) \mid X_2 = F_2^{-1}(v)\}$ for a bivariate t-distribution with correlation coefficients $\rho = -0.8, -0.6, \ldots, 0.6, 0.8$.

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function F to its corresponding one-dimensional marginal distribution functions $F_i,\ i = 1, \ldots, n$, in the following way:

$$F(x_1, \ldots, x_n) = C\{F_1(x_1), \ldots, F_n(x_n)\}.$$

We refer the reader to Joe (1997), Nelsen (1999) or Härdle, Kleinow, and Stahl (2002) for more information on copulae. The following representation shows that tail dependence is a copula property. Thus, many copula features transfer to the tail-dependence coefficient such as the invariance under strictly increasing transformations of the margins. If X is a continuous bivariate random vector, then straightforward calculation yields:

$$\lambda_U = \lim_{v \to 1^-} \frac{1 - 2v + C(v,v)}{1 - v}, \tag{1.4}$$

where C denotes the copula of X. Analogously, $\lambda_L = \lim_{v \to 0^+} \frac{C(v,v)}{v}$ holds for the lower tail-dependence coefficient.

1.3 Calculation of the tail-dependence coefficient

1.3.1 Archimedean copulae

Archimedean copulae form an important class of copulae which are easy to construct and have good analytical properties. A bivariate Archimedean copula has the form $C(u,v) = \psi^{[-1]}\{\psi(u) + \psi(v)\}$ for some continuous, strictly decreasing, and convex *generator* function $\psi : [0,1] \to [0,\infty]$ such that $\psi(1) = 0$ and the pseudo-inverse function $\psi^{[-1]}$ is defined by:

$$\psi^{[-1]}(t) = \begin{cases} \psi^{-1}(t), & 0 \le t \le \psi(0), \\ 0, & \psi(0) < t \le \infty. \end{cases}$$

We call ψ *strict* if $\psi(0) = \infty$. In that case $\psi^{[-1]} = \psi^{-1}$. Within the framework of tail dependence for Archimedean copulae, the following result can be shown (Schmidt, 2003). Note that the one-sided derivatives of ψ exist, as ψ is a convex function. In particular, $\psi'(1)$ and $\psi'(0)$ denote the one-sided derivatives at the boundary of the domain of ψ. Then:

i) upper tail-dependence implies $\psi'(1) = 0$ and $\lambda_U = 2 - (\psi^{-1} \circ 2\psi)'(1)$,

ii) $\psi'(1) < 0$ implies upper tail-independence,

iii) $\psi'(0) > -\infty$ or a non-strict ψ implies lower tail-independence,

iv) lower tail-dependence implies $\psi'(0) = -\infty$, a strict ψ, and $\lambda_L = (\psi^{-1} \circ 2\psi)'(0)$.

Tables 1.1 and 1.2 list various Archimedean copulae in the same ordering as provided in Nelsen (1999, Table 4.1, p. 94) and in Härdle, Kleinow, and Stahl (2002, Table 2.1, p. 42) and the corresponding upper and lower tail-dependence coefficients (TDCs).

Table 1.1: Various selected Archimedean copulae. The numbers in the first column correspond to the numbers of Table 4.1 in Nelsen (1999), p. 94.

Number & Type		$C(u,v)$	Parameters
(1)	Clayton	$\max\left\{(u^{-\theta}+v^{-\theta}-1)^{-1/\theta},0\right\}$	$\theta \in [-1,\infty)\backslash\{0\}$
(2)		$\max\left[1-\left\{(1-u)^{\theta}+(1-v)^{\theta}\right\}^{1/\theta},0\right]$	$\theta \in [1,\infty)$
(3)	Ali-Mikhail-Haq	$\dfrac{uv}{1-\theta(1-u)(1-v)}$	$\theta \in [-1,1)$
(4)	Gumbel-Hougaard	$\exp\left[-\left\{(-\log u)^{\theta}+(-\log v)^{\theta}\right\}^{1/\theta}\right]$	$\theta \in [1,\infty)$
(12)		$\left[1+\left\{(u^{-1}-1)^{\theta}+(v^{-1}-1)^{\theta}\right\}^{1/\theta}\right]^{-1}$	$\theta \in [1,\infty)$
(14)		$\left[1+\left\{(u^{-1/\theta}-1)^{\theta}+(v^{-1/\theta}-1)^{\theta}\right\}^{1/\theta}\right]^{-\theta}$	$\theta \in [1,\infty)$
(19)		$\theta/\log\left(e^{\theta/u}+e^{\theta/v}-e^{\theta}\right)$	$\theta \in (0,\infty)$

1.3.2 Elliptically-contoured distributions

In this section, we calculate the tail-dependence coefficient for elliptically-contoured distributions (briefly: elliptical distributions). Well-known elliptical distributions are the multivariate normal distribution, the multivariate t-distribution, the multivariate logistic distribution, the multivariate symmetric stable distribution, and the multivariate symmetric generalized-hyperbolic distribution.

Elliptical distributions are defined as follows: let X be an n-dimensional random vector and $\Sigma \in \mathbb{R}^{n\times n}$ be a symmetric positive semi-definite matrix. If $X-\mu$, for some $\mu \in \mathbb{R}^n$, possesses a characteristic function of the form $\phi_{X-\mu}(t) = \Psi(t^\top \Sigma t)$ for some function $\Psi : \mathbb{R}_0^+ \to \mathbb{R}$, then X is said to be

Table 1.2: Tail-dependence coefficients (TDCs) and generators ψ_θ for various selected Archimedean copulae. The numbers in the first column correspond to the numbers of Table 4.1 in Nelsen (1999), p. 94.

Number & Type	$\psi_\theta(t)$	Parameter θ	Upper-TDC	Lower-TDC
(1) Pareto	$t^{-\theta}-1$	$[-1,\infty)\backslash\{0\}$	0 for $\theta > 0$	$2^{-1/\theta}$ for $\theta > 0$
(2)	$(1-t)^\theta$	$[1,\infty)$	$2-2^{1/\theta}$	0
(3) Ali-Mikhail-Haq	$\log\dfrac{1-\theta(1-t)}{t}$	$[-1,1)$	0	0
(4) Gumbel-Hougaard	$(-\log t)^\theta$	$[1,\infty)$	$2-2^{1/\theta}$	0
(12)	$\left(\frac{1}{t}-1\right)^\theta$	$[1,\infty)$	$2-2^{1/\theta}$	$2^{-1/\theta}$
(14)	$\left(t^{-1/\theta}-1\right)^\theta$	$[1,\infty)$	$2-2^{1/\theta}$	$\frac{1}{2}$
(19)	$e^{\theta/t}-e^\theta$	$(0,\infty)$	0	1

elliptically distributed with parameters μ (location), Σ (dispersion), and Ψ. Let $E_n(\mu,\Sigma,\Psi)$ denote the class of elliptically-contoured distributions with the latter parameters. We call Ψ the *characteristic generator*. The density function, if it exists, of an elliptically-contoured distribution has the following form:

$$f(x) = |\Sigma|^{-1/2} g\left\{(x-\mu)^\top \Sigma^{-1}(x-\mu)\right\}, \quad x \in \mathbb{R}^n, \tag{1.5}$$

for some function $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, which we call the *density generator*.

Observe that the name "elliptically-contoured distribution" is related to the elliptical contours of the latter density. For a more detailed treatment of elliptical distributions see the monograph of Fang, Kotz, and Ng (1990) or Cambanis, Huang, and Simon (1981).

In connection with financial applications, Bingham and Kiesel (2002) and Bingham, Kiesel, and Schmidt (2002) propose a semi-parametric approach for elliptical distributions by estimating the *parametric* component (μ, Σ) separately from the density generator g. In their setting, the density generator is estimated by means of a *nonparametric* statistics.

Schmidt (2002b) shows that bivariate elliptically-contoured distributions are upper and lower tail-dependent if the tail of their density generator is regularly varying, i.e. the tail behaves asymptotically like a power function. Further, a necessary condition for tail dependence is given which is more general than regular variation of the latter tail: more precisely, the tail must be O-regularly varying (see Bingham, Goldie, and Teugels (1987) for the definition of O-regular variation). Although the equivalence of tail dependence and regularly-varying density generator has not been shown, all density generators of well-known elliptical distributions possess either a regularly-varying tail or a not O-regularly-varying tail. This justifies a restriction to the class of elliptical distributions with regularly-varying density generator if tail dependence is required. In particular, tail dependence is solely determined by the tail behavior of the density generator (except for completely correlated random variables which are always tail dependent).

The following closed-form expression exists (Schmidt, 2002b) for the upper and lower tail-dependence coefficient of an elliptically-contoured random vector $(X_1, X_2)^\top \in E_2(\mu, \Sigma, \Psi)$ with positive-definite matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{11} & \sigma_{12} \end{pmatrix},$$

having a regularly-varying density generator g with regular variation index $-\alpha/2 - 1 < 0$:

$$\lambda \stackrel{\text{def}}{=} \lambda_U = \lambda_L = \frac{\displaystyle\int_0^{h(\rho)} \frac{u^\alpha}{\sqrt{1-u^2}} du}{\displaystyle\int_0^1 \frac{u^\alpha}{\sqrt{1-u^2}} du}, \tag{1.6}$$

where $\rho \stackrel{\text{def}}{=} \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$ and $h(\rho) \stackrel{\text{def}}{=} \left(1 + \frac{(1-\rho)^2}{1-\rho^2}\right)^{-1/2}$.

Note that ρ corresponds to the "correlation" coefficient when it exists (Fang, Kotz, and Ng, 1990). Moreover, the upper tail-dependence coefficient λ_U coincides with the lower tail-dependence coefficient λ_L and depends only on the "correlation" coefficient ρ and the regular variation index α, see Figure 1.3.

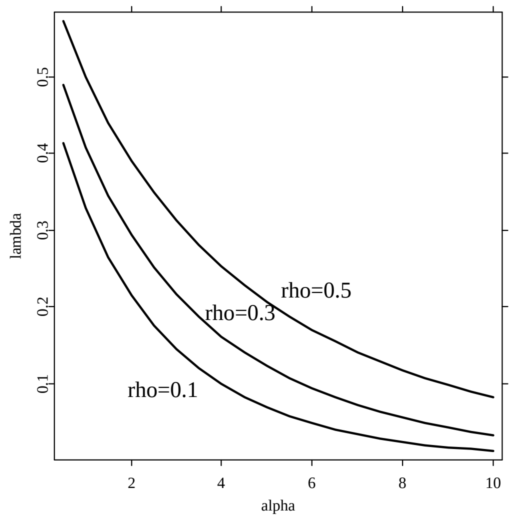

Figure 1.3: Tail-dependence coefficient λ versus regular variation index α for "correlation" coefficients $\rho = 0.5,\ 0.3,\ 0.1$.

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Table 1.3 lists various elliptical distributions, the corresponding density generators (here c_n denotes a normalizing constant depending only on the dimension n) and the associated regular variation index α from which one easily derives the tail-dependence coefficient using formula (1.6).

Table 1.3: Tail index α for various density generators g of multivariate elliptical distributions. K_ν denotes the modified Bessel function of the third kind (or Macdonald function).

Number & Type	Density generator g or characteristic generator Ψ	Parameters	α for $n=2$
(23) Normal	$g(u) = c_n \exp(-u/2)$	—	∞
(24) t	$g(u) = c_n \left(1 + \frac{t}{\theta}\right)^{-(n+\theta)/2}$	$\theta > 0$	θ
(25) Symmetric general. hyperbolic	$g(u) = c_n \frac{K_{\lambda-\frac{n}{2}}\{\sqrt{\varsigma(\chi+u)}\}}{(\sqrt{\chi+u})^{\frac{n}{2}-\lambda}}$	$\varsigma, \chi > 0$ $\lambda \in \mathbb{R}$	∞
(26) Symmetric θ-stable	$\Psi(u) = \exp\left\{-\left(\frac{1}{2}u\right)^{\theta/2}\right\}$	$\theta \in (0, 2]$	θ
(27) logistic	$g(u) = c_n \frac{\exp(-u)}{\{1+\exp(-u)\}^2}$	—	∞

1.3.3 Other copulae

For many other closed form copulae one can explicitly derive the tail-dependence coefficient. Tables 1.4 and 1.5 list some well-known copula functions and the corresponding lower and upper TDCs.

Table 1.4: Various copulae. Copulae BBx are provided in Joe (1997).

Number & Type	$C(u,v)$	Parameters
(28) Raftery	$g\{\min(u,v),\max(u,v);\theta\}$ with $g\{x,y;\theta\} = x - \frac{1-\theta}{1+\theta} x^{1/(1-\theta)} \left(y^{-\theta/(1-\theta)} - y^{1/(1-\theta)} \right)$	$\theta \in [0,1]$
(29) BB1	$\left[1 + \left\{ (u^{-\theta}-1)^{\delta} + (v^{-\theta}-1)^{\delta} \right\}^{1/\delta} \right]^{-1/\theta}$	$\theta \in (0,\infty)$ $\delta \in [1,\infty)$
(30) BB4	$\left[u^{-\theta} + v^{-\theta} - 1 - \right.$ $\left. - \left\{ (u^{-\theta}-1)^{-\delta} + (v^{-\theta}-1)^{-\delta} \right\}^{-1/\delta} \right]^{-1/\theta}$	$\theta \in [0,\infty)$ $\delta \in (0,\infty)$
(31) BB7	$1 - \left(1 - \left[\left\{ 1-(1-u)^{\theta} \right\}^{-\delta} + \right. \right.$ $\left. \left. + \left\{ 1-(1-v)^{\theta} \right\}^{-\delta} - 1 \right]^{-1/\delta} \right)^{1/\theta}$	$\theta \in [1,\infty)$ $\delta \in (0,\infty)$
(32) BB8	$\frac{1}{\delta} \left(1 - \left[1 - \left\{ 1-(1-\delta)^{\theta} \right\}^{-1} \cdot \right. \right.$ $\left. \left. \cdot \left\{ 1-(1-\delta u)^{\theta} \right\} \left\{ 1-(1-\delta v)^{\theta} \right\} \right]^{1/\theta} \right)$	$\theta \in [1,\infty)$ $\delta \in [0,1]$
(33) BB11	$\theta \min(u,v) + (1-\theta)uv$	$\theta \in [0,1]$
(34) C_{Ω} in Junker et al. (2002)	$\beta C^s_{(\bar\theta,\bar\delta)}(u,v) - (1-\beta) C_{(\theta,\delta)}(u,v)$ with Archim. generator $\psi_{(\theta,\delta)}(t) = \left(-\log \frac{e^{-\theta t}-1}{e^{-\theta}-1} \right)^{\delta}$ $C^s_{(\bar\theta,\bar\delta)}$ is the survival copula with param. $(\bar\theta,\bar\delta)$	$\theta, \bar\theta \in \mathbb{R}\backslash\{0\}$ $\delta, \bar\delta \geq 1$ $\beta \in [0,1]$

1.4 Estimating the tail-dependence coefficient

Suppose $X,\ X^{(1)},\ldots,X^{(m)}$ are i.i.d. bivariate random vectors with distribution function F and copula C. We assume continuous marginal distribution functions $F_i,\ i=1,2$. Tests for tail dependence or tail independence are given for example in Ledford and Tawn (1996) or Draisma et al. (2004).

We consider the following three (non-)parametric estimators for the lower and upper tail-dependence coefficients λ_U and λ_L. These estimators have been discussed in Huang (1992) and Schmidt and Stadtmüller (2003). Let C_m be the

Table 1.5: Tail-dependence coefficients (TDCs) for various copulae. Copulae BBx are provided in Joe (1997).

Number & Type	Parameters	upper-TDC	lower-TDC
(28) Raftery	$\theta \in [0,1]$	0	$\frac{2\theta}{1+\theta}$
(29) BB1	$\theta \in (0,\infty)$ $\delta \in [1,\infty)$	$2-2^{1/\delta}$	$2^{-1/(\theta\delta)}$
(30) BB4	$\theta \in [0,\infty)$ $\delta \in (0,\infty)$	$2^{-1/\delta}$	$(2-$ $2^{-1/\delta})^{-1/\theta}$
(31) BB7	$\theta \in [1,\infty)$ $\delta \in (0,\infty)$	$2-2^{1/\theta}$	$2^{-1/\delta}$
(32) BB8	$\theta \in [1,\infty)$ $\delta \in [0,1]$	$2-$ $-2(1-\delta)^{\theta-1}$	0
(33) BB11	$\theta \in [0,1]$	θ	θ
(34) C_Ω in Junker et al. (2002)	$\theta,\bar{\theta} \in \mathbb{R}\backslash\{0\}$ $\delta,\bar{\delta} \geq 1$ $\beta \in [0,1]$	$(1-\beta)\cdot$ $\cdot(2-2^{1/\delta})$	$\beta(2-2^{1/\bar{\delta}})$

empirical copula defined by:

$$C_m(u,v) = F_m(F_{1m}^{-1}(u), F_{2m}^{-1}(v)), \tag{1.7}$$

with F_m and F_{im} denoting the empirical distribution functions corresponding to F and F_i, $i = 1,2$, respectively. Let $R_{m1}^{(j)}$ and $R_{m2}^{(j)}$ be the rank of $X_1^{(j)}$ and $X_2^{(j)}$, $j = 1,\ldots,m$, respectively. The first estimators are based on formulas (1.1) and (1.2):

$$\begin{aligned}\hat{\lambda}^{(1)}_{U,m} &= \frac{m}{k} C_m\Big(\Big(1-\frac{k}{m},1\Big]\times\Big(1-\frac{k}{m},1\Big]\Big)\\ &= \frac{1}{k}\sum_{j=1}^{m} I(R^{(j)}_{m1} > m-k, R^{(j)}_{m2} > m-k) \qquad (1.8)\end{aligned}$$

and

$$\hat{\lambda}^{(1)}_{L,m} = \frac{m}{k} C_m\Big(\frac{k}{m},\frac{k}{m}\Big) = \frac{1}{k}\sum_{j=1}^{m} I(R^{(j)}_{m1} \le k, R^{(j)}_{m2} \le k), \qquad (1.9)$$

where $k = k(m) \to \infty$ and $k/m \to 0$ as $m \to \infty$, and the first expression in (1.8) has to be understood as the empirical copula-measure of the interval $(1-k/m, 1] \times (1-k/m, 1]$. The second type of estimator is already well known in multivariate extreme-value theory (Huang, 1992). We only provide the estimator for the upper TDC:

$$\begin{aligned}\hat{\lambda}^{(2)}_{U,m} &= 2-\frac{m}{k}\Big\{1-C_m\Big(1-\frac{k}{m},1-\frac{k}{m}\Big)\Big\}\\ &= 2-\frac{1}{k}\sum_{j=1}^{m} I(R^{(j)}_{m1} > m-k \text{ or } R^{(j)}_{m2} > m-k), \qquad (1.10)\end{aligned}$$

with $k = k(m) \to \infty$ and $k/m \to 0$ as $m \to \infty$. The optimal choice of k is related to the usual variance-bias problem and we refer the reader to Peng (1998) for more details. Strong consistency and asymptotic normality for both types of nonparametric estimators are also addressed in the latter three reference.

Now we focus on an elliptically-contoured bivariate random vector X. In the presence of tail dependence, previous arguments justify a sole consideration of elliptical distributions having a regularly-varying density generator with regular variation index α. This implies that the distribution function of $||X||_2$ has also a regularly-varying tail with index α (Schmidt, 2002b). Formula (1.6) shows that the upper and lower tail-dependence coefficients λ_U and λ_L depend only on the regular variation index α and the "correlation" coefficient ρ. Hence, we propose the following parametric estimator for λ_U and λ_L:

$$\hat{\lambda}^{(3)}_{U,m} = \hat{\lambda}^{(3)}_{L,m} = \lambda^{(3)}_U(\hat{\alpha}_m, \hat{\rho}_m). \qquad (1.11)$$

Several robust estimators $\hat{\rho}_m$ for ρ are provided in the literature such as estimators based on techniques of multivariate trimming (Hahn, Mason, and Weiner, 1991), minimum-volume ellipsoid estimators (Rousseeuw and van Zomeren, 1990), and least square estimators (Frahm et al., 2002). The Hill estimator which serves as an estimator for the regular variation index α has already been considered in Chapter **??**.

For more details regarding the relationship between the regular variation index α, the density generator, and the random variable $||X||_2$ we refer to Schmidt (2002b). Observe that even though the estimator for the regular variation index α might be unbiased, the TDC estimator $\hat{\lambda}_{U,m}^{(3)}$ is biased due to the integral transform.

1.5 Comparison of TDC estimators

In this section we investigate the finite-sample properties of the introduced TDC estimators. One thousand independent copies of $m = 500, 1000$, and 2000 i.i.d. random vectors (m denotes the sample length) of a bivariate standard t-distribution with $\theta = 1.5, 2$, and 3 degrees of freedom are generated and the upper TDCs are estimated. Note that the parameter θ equals the regular variation index α which we discussed in the context of elliptically-contoured distributions. The empirical bias and root-mean-squared error (RMSE) for all three introduced TDC estimation methods are derived and presented in Tables 1.6, 1.7, and 1.8, respectively.

Table 1.6: Bias and RMSE for the nonparametric upper TDC estimator $\hat{\lambda}_U^{(1)}$ (multiplied by 10^3). The sample length is denoted by m.

Original parameters	$\theta = 1.5$ $\lambda_U = 0.2296$	$\theta = 2$ $\lambda_U = 0.1817$	$\theta = 3$ $\lambda_U = 0.1161$
Estimator	$\hat{\lambda}_U^{(1)}$ Bias (RMSE)	$\hat{\lambda}_U^{(1)}$ Bias (RMSE)	$\hat{\lambda}_U^{(1)}$ Bias (RMSE)
$m = 500$	25.5 (60.7)	43.4 (72.8)	71.8 (92.6)
$m = 1000$	15.1 (47.2)	28.7 (55.3)	51.8 (68.3)
$m = 2000$	8.2 (38.6)	19.1 (41.1)	36.9 (52.0)

Table 1.7: Bias and RMSE for the nonparametric upper TDC estimator $\hat{\lambda}_U^{(2)}$ (multiplied by 10^3). The sample length is denoted by m.

Original parameters	$\theta = 1.5$ $\lambda_U = 0.2296$	$\theta = 2$ $\lambda_U = 0.1817$	$\theta = 3$ $\lambda_U = 0.1161$
Estimator	$\hat{\lambda}_U^{(2)}$ Bias (RMSE)	$\hat{\lambda}_U^{(2)}$ Bias (RMSE)	$\hat{\lambda}_U^{(2)}$ Bias (RMSE)
$m = 500$	53.9 (75.1)	70.3 (88.1)	103.1 (116.4)
$m = 1000$	33.3 (54.9)	49.1 (66.1)	74.8 (86.3)
$m = 2000$	22.4 (41.6)	32.9 (47.7)	56.9 (66.0)

Table 1.8: Bias and RMSE for the parametric upper TDC estimator $\hat{\lambda}_U^{(3)}$ (multiplied by 10^3). The sample length is denoted by m.

Original parameters	$\theta = 1.5$ $\lambda_U = 0.2296$	$\theta = 2$ $\lambda_U = 0.1817$	$\theta = 3$ $\lambda_U = 0.1161$
Estimator	$\hat{\lambda}_U^{(3)}$ Bias (RMSE)	$\hat{\lambda}_U^{(3)}$ Bias (RMSE)	$\hat{\lambda}_U^{(3)}$ Bias (RMSE)
$m = 500$	1.6 (30.5)	3.5 (30.8)	16.2 (33.9)
$m = 1000$	2.4 (22.4)	5.8 (23.9)	15.4 (27.6)
$m = 2000$	2.4 (15.5)	5.4 (17.0)	12.4 (21.4)

Regarding the parametric approach we apply the procedure introduced in Section 1.4 and estimate ρ by a trimmed empirical correlation coefficient with trimming proportion 0.05% and α $(= \theta)$ by a Hill estimator. For the latter we choose the optimal threshold value k according to Drees and Kaufmann (1998). The empirical bias and RMSE corresponding to the estimation of ρ and α are provided in Tables 1.9 and 1.10. Observe that Pearson's correlation coefficient ρ does not exist for $\theta < 2$. In this case, ρ denotes some dependence parameter and a more robust estimation procedure should be used (Frahm et al., 2002).

Table 1.9: Bias and RMSE for the estimator of the regular variation index α (multiplied by 10^3). The sample length is denoted by m.

Original parameters	$\theta = 1.5$ $\alpha = 1.5$	$\theta = 2$ $\alpha = 2$	$\theta = 3$ $\alpha = 3$
Estimator	$\hat{\alpha}$ Bias (RMSE)	$\hat{\alpha}$ Bias (RMSE)	$\hat{\alpha}$ Bias (RMSE)
$m = 500$	2.2 (211.9)	−19.8 (322.8)	−221.9 (543.7)
$m = 1000$	−14.7 (153.4)	−48.5 (235.6)	−242.2 (447.7)
$m = 2000$	−15.7 (101.1)	−60.6 (173.0)	−217.5 (359.4)

Table 1.10: Bias and RMSE for the "correlation" coefficient estimator $\hat{\rho}$ (multiplied by 10^3). The sample length is denoted by m.

Original parameters	$\theta = 1.5$ $\rho = 0$	$\theta = 2$ $\rho = 0$	$\theta = 3$ $\rho = 0$
Estimator	$\hat{\rho}$ Bias (RMSE)	$\hat{\rho}$ Bias (RMSE)	$\hat{\rho}$ Bias (RMSE)
$m = 500$	0.02 (61.6)	−2.6 (58.2)	2.1 (56.5)
$m = 1000$	−0.32 (44.9)	1.0 (42.1)	0.6 (39.5)
$m = 2000$	0.72 (32.1)	−1.2 (29.3)	−1.8 (27.2)

Finally, Figures 1.4 and 1.5 illustrate the (non-)parametric estimation results of the upper TDC estimator $\hat{\lambda}_U^{(i)}$, $i = 1, 2, 3$. Presented are 3×1000 TDC estimations with sample lengths $m = 500, 1000$ and 2000. The plots visualize the decreasing empirical bias and variance for increasing sample length.

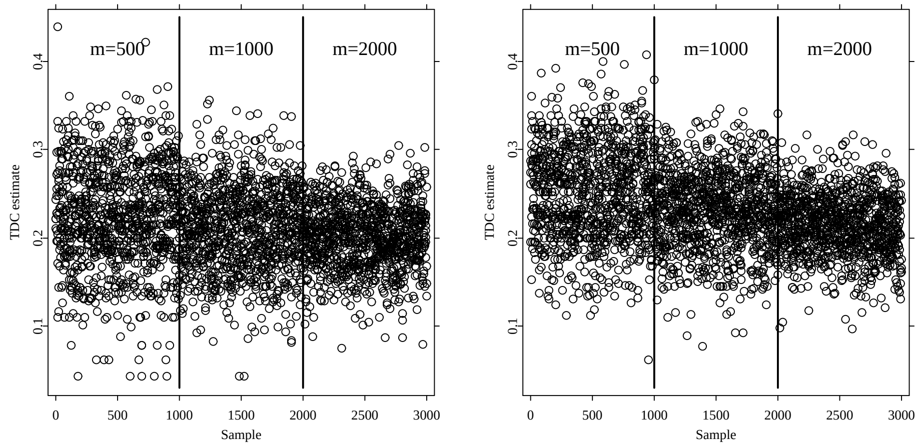

Figure 1.4: Nonparametric upper TDC estimates $\hat{\lambda}_U^{(1)}$ (left panel) and $\hat{\lambda}_U^{(2)}$ (right panel) for 3×1000 i.i.d. samples of size $m = 500, 1000, 2000$ from a bivariate t-distribution with parameters $\theta = 2$, $\rho = 0$, and $\lambda_U^{(1)} = \lambda_U^{(2)} = 0.1817$.

STFtail04.xpl

The empirical study shows that the TDC estimator $\hat{\lambda}_U^{(3)}$ outperforms the other two estimators. For $m = 2000$, the bias (RMSE) of $\hat{\lambda}_U^{(1)}$ is three (two and a half) times larger than the bias (RMSE) of $\hat{\lambda}_U^{(3)}$, whereas the bias (RMSE) of $\hat{\lambda}_U^{(2)}$ is two (ten percent) times larger than the bias (RMSE) of $\hat{\lambda}_U^{(1)}$. More empirical and statistical results regarding the estimators $\hat{\lambda}_U^{(1)}$ and $\hat{\lambda}_U^{(2)}$ are given in Schmidt and Stadtmüller (2003). However, note that the estimator $\hat{\lambda}_U^{(3)}$ was especially developed for bivariate elliptically-contoured distributions. Thus, the estimator $\hat{\lambda}_U^{(1)}$ is recommended for TDC estimations of non-elliptical or unknown bivariate distributions.

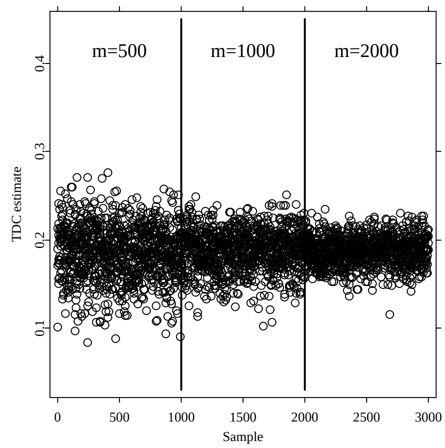

Figure 1.5: Nonparametric upper TDC estimates $\hat{\lambda}_U^{(3)}$ for 3×1000 i.i.d. samples of size $m = 500, 1000, 2000$ from a bivariate t-distribution with parameters $\theta = 2$, $\rho = 0$, and $\lambda_U^{(3)} = 0.1817$. STFtail05.xpl

1.6 Tail dependence of asset and FX returns

Tail dependence is indeed often found in financial data series. Consider two scatter plots of daily negative log-returns of a tuple of financial securities and the corresponding upper TDC estimate $\hat{\lambda}_U^{(1)}$ for various k (for notational convenience we drop the index m).

The first data set (D_1) contains negative daily stock log-returns of BMW and Deutsche Bank for the time period 1992-2001. The second data set (D_2) consists of negative daily exchange rate log-returns of DEM/USD and JPY/USD (so-called FX returns) for the time period 1989-2001. For modelling reasons we assume that the daily log-returns are i.i.d. observations. Figures 1.6 and 1.7 show the presence of tail dependence and the order of magnitude of the tail-dependence coefficient. Tail dependence is present if the plot of TDC estimates $\hat{\lambda}_U^{(1)}$ against the thresholds k shows a characteristic plateau for small k. The existence of this plateau for tail-dependent distributions is justified by a regular variation property of the tail distribution; we refer the reader to Peng (1998) or Schmidt and Stadtmüller (2003) for more details. By contrast, the

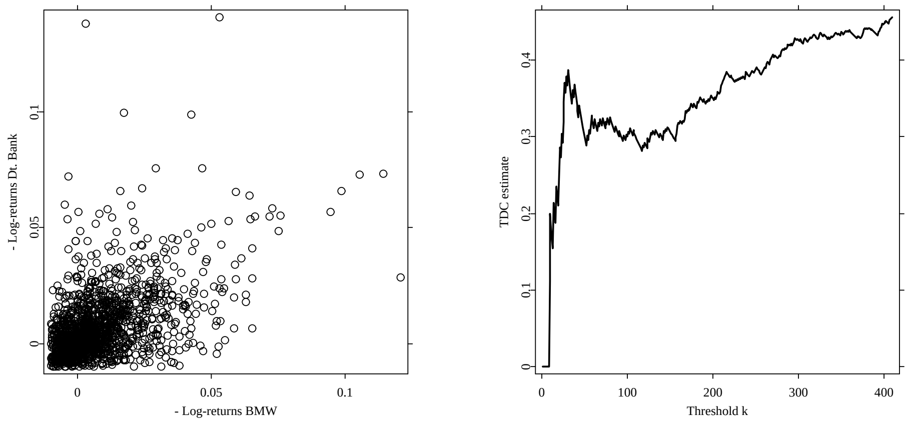

Figure 1.6: Scatter plot of BMW versus Deutsche Bank negative daily stock log-returns (2347 data points) and the corresponding TDC estimate $\hat{\lambda}_U^{(1)}$ for various thresholds k.

STFtail06.xpl

characteristic plateau is not observable if the distribution is tail independent.

The typical variance-bias problem for various thresholds k can be also observed in Figures 1.6 and 1.7. In particular, a small k comes along with a large variance of the TDC estimator, whereas increasing k results in a strong bias. In the presence of tail dependence, k is chosen such that the TDC estimate $\hat{\lambda}_U^{(1)}$ lies on the plateau between the decreasing variance and the increasing bias. Thus for the data set D_1 we take k between 80 and 110 which provides a TDC estimate of $\hat{\lambda}_{U,D_1}^{(1)} = 0.31$, whereas for D_2 we choose k between 40 and 90 which yields $\hat{\lambda}_{U,D_2}^{(1)} = 0.17$.

The importance of the detection and the estimation of tail dependence becomes clear in the next section. In particular, we show that the Value at Risk estimation of a portfolio is closely related to the concept of tail dependence. A proper analysis of tail dependence results in an adequate choice of the portfolio's loss distribution and leads to a more precise assessment of the Value at Risk .

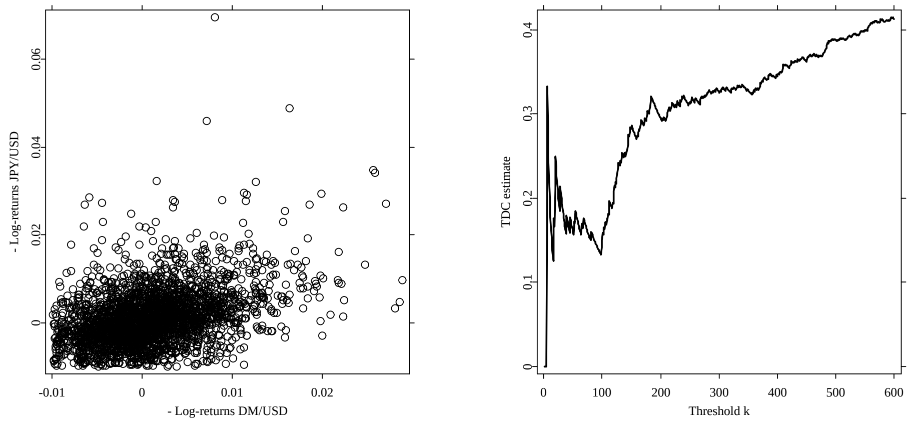

Figure 1.7: Scatter plot of DEM/USD versus JPY/USD negative daily exchange rate log-returns (3126 data points) and the corresponding TDC estimate $\hat{\lambda}_U^{(1)}$ for various thresholds k. STFtail07.xpl

1.7 Value at Risk – A simulation study

Value at Risk (VaR) estimations refer to the estimation of high target quantiles of single asset or portfolio loss distributions. Thus, VaR estimations are very sensitive towards the tail behavior of the underlying distribution model.

On the one hand, the VaR of a portfolio is affected by the tail distribution of each single asset. On the other hand, the general dependence structure and especially the tail-dependence structure among all assets have a strong impact on the portfolio's VaR, too. With the concept of tail dependence, we supply a methodology for measuring and modelling one particular type of dependence of extreme events.

What follows, provides empirical justification that the portfolio's VaR estimation depends heavily on a proper specification of the (tail-)dependence structure of the underlying asset-return vector. To illustrate our assertion we consider three financial data sets: the first two data sets D_1 and D_2 refer again to the daily stock log-returns of BMW and Deutsche Bank for the time period 1992-

2001 and the daily exchange rate log-returns of DEM/USD and JPY/USD for the time period 1989-2001, respectively. The third data set (D_3) contains exchange rate log-returns of FFR/USD and DEM/USD for the time period 1982-2002.

Typically, in practice, either a multivariate normal distribution or multivariate t-distribution is fitted to the data in order to describe the random behavior (market riskiness) of asset returns. Especially multivariate t-distributions have recently gained the attraction of practitioners due to their ability to model heavy tails while still having the advantage of being in the class of elliptically-contoured distributions. Recall that the multivariate normal distribution has thin tailed marginals which exhibit no tail-dependence, and the t-distribution possesses heavy tailed marginals which are tail dependent (see Section 1.3.2). Due to the different tail behavior, one might pick one of the latter two distribution classes if the data are elliptically contoured. However, elliptically-contoured distributions require a very strong symmetry of the data and might not be appropriate in many circumstances.

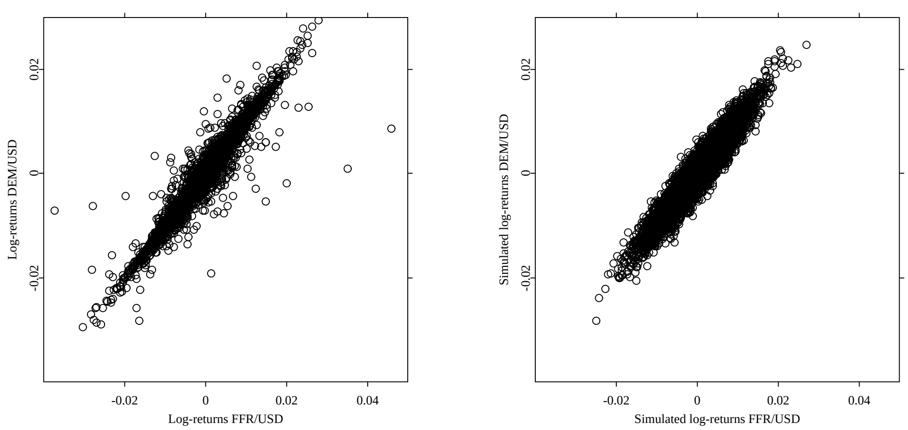

Figure 1.8: Scatter plot of foreign exchange data (left panel) and simulated normal pseudo-random variables (right panel) of FFR/USD versus DEM/USD negative daily exchange rate log-returns (5189 data points).

STFtail08.xpl

For example, the scatter plot of the data set D_3 in Figure 1.8 reveals that its distributional structure does not seem to be elliptically contoured at all. To circumvent this problem, one could fit a distribution from a broader distribution class, such as a generalized hyperbolic distribution (Eberlein and Keller, 1995; Bingham and Kiesel, 2002). Alternatively, a split of the dependence structure and the marginal distribution functions via the theory of copulae (as described in Section 1.2) seems to be also attractive. This split exploits the fact that statistical (calibration) methods are well established for one-dimensional distribution functions.

For the data sets D_1, D_2, and D_3, one-dimensional t-distributions are utilized to model the marginal distributions. The choice of an appropriate copula function turns out to be delicate. Two structural features are important in the context of VaR estimations regarding the choice of the copula. First, the general structure (symmetry) of the chosen copula should coincide with the dependence structure of the real data. We visualize the dependence structure of the sample data via the respective empirical copula (Figure 1.9), i.e. the marginal distributions are standardized by the corresponding empirical distribution functions. Second, if the data show tail dependence than one must utilize a copula which comprises tail dependence. Especially VaR estimations at a small confidence level are very sensitive towards tail dependence. Figure 1.9 indicates that the FX data set D_3 has significantly more dependence in the lower tail than the simulated data from a fitted bivariate normal copula. The data clustering in the lower left corner of the scatter plot of the empirical copula is a strong indication for tail dependence.

Based on the latter findings, we use a t-copula (which allows for tail dependence, see Section 1.3.2) and t-distributed marginals (which are heavy tailed). Note, the resulting common distribution is only elliptically contoured if the degrees of freedom of the t-copula and the t-margins coincide, since in this case the common distribution corresponds to a multivariate t-distribution. The parameters of the marginals and the copula are separately estimated in two consecutive steps via maximum likelihood. For statistical properties of the latter procedure, which is called Inference Functions for Margins method (IFM), we refer to Joe and Xu (1996).

Tables 1.11, 1.12, and 1.13 compare the historical VaR estimates of the data sets D_1, D_2, and D_3 with the average of 100 VaR estimates which are simulated from different distributions. The fitted distribution is either a bivariate normal, a bivariate t-distribution or a bivariate distribution with t-copula and t-marginals. The respective standard deviation of the VaR estimations are provided in parenthesis. For a better exposition, we have multiplied all numbers by 10^5.

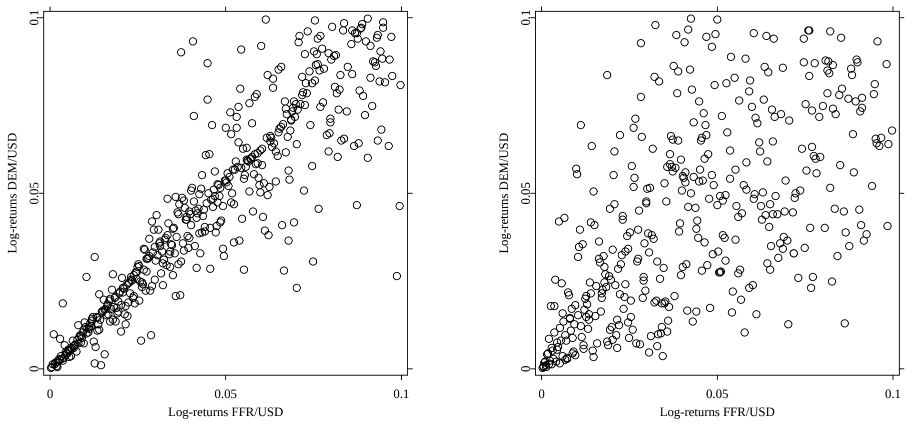

Figure 1.9: Lower left corner of the empirical copula density plots of real data (left panel) and simulated normal pseudo-random variables (right panel) of FFR/USD versus DEM/USD negative daily exchange rate log-returns (5189 data points). STFtail09.xpl

Table 1.11: Mean and standard deviation of 100 VaR estimations (multiplied by 10^5) from simulated data following different distributions which are fitted to the data set D_1.

Quantile	Historical VaR	Normal distribution		t-distribution		t-copula & t-marginals	
		Mean	(Std)	Mean	(Std)	Mean	(Std)
0.01	489.93	397.66	(13.68)	464.66	(39.91)	515.98	(36.54)
0.025	347.42	335.28	(9.67)	326.04	(18.27)	357.40	(18.67)
0.05	270.41	280.69	(7.20)	242.57	(10.35)	260.27	(11.47)

Table 1.12: Mean and standard deviation of 100 VaR estimations (multiplied by 10^5) from simulated data following different distributions which are fitted to the data set D_2.

Quantile	Historical VaR	Normal distribution	t-distribution	t-copula & t-marginals
		Mean (Std)	Mean (Std)	Mean (Std)
0.01	155.15	138.22 (4.47)	155.01 (8.64)	158.25 (8.24)
0.025	126.63	116.30 (2.88)	118.28 (4.83)	120.08 (4.87)
0.05	98.27	97.56 (2.26)	92.35 (2.83)	94.14 (3.12)

Table 1.13: Mean and standard deviation of 100 VaR estimations (multiplied by 10^5) from simulated data following different distributions which are fitted to the data set D_3.

Quantile	Historical VaR	Normal distribution	t-distribution	t-copula & t-marginals
		Mean (Std)	Mean (Std)	Mean (Std)
0.01	183.95	156.62 (3.65)	179.18 (9.75)	179.41 (6.17)
0.025	141.22	131.54 (2.41)	124.49 (4.43)	135.21 (3.69)
0.05	109.94	110.08 (2.05)	91.74 (2.55)	105.67 (2.59)

The results of the latter tables clearly show that the fitted bivariate normal-distribution does not yield an overall satisfying estimation of the VaR for all data sets D_1, D_2, and D_3. The poor estimation results for the $0.01-$ and $0.025-$quantile VaR (i.e. the mean of the VaR estimates deviate strongly from the historical VaR estimate) are mainly caused by the thin tails of the normal distribution. By contrast, the bivariate t-distribution provides good estimations of the historical VaR for the data sets D_1 and D_2 over all quantiles. However, both data sets are approximately elliptically-contoured distributed since the estimated parameters of the copula and the marginals are almost equal. For example for the data set D_1, the estimated degree of freedom of the t-copula is 3.05 whereas the estimated degrees of freedom of the t-marginals are 2.99

and 3.03, respectively. We have already discussed that the distribution of the data set D_3 is not elliptically contoured. Indeed, the VaR estimation improves with a splitting of the copula and the marginals. The corresponding estimated degree of freedom of the t-copula is 1.11 whereas the estimated degrees of freedom of the t-marginals are 4.63 and 5.15. Finally, note that the empirical standard deviations do significantly differ between the VaR estimation based on the multivariate t-distribution and the t-copula, respectively.

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