A Non-parametric Test Frameworks

A.1 Testing Mean-Variance-Stability

Using partial sums of sample moments to test for constant correlation has been suggested by Wied et al. [2012b], who derive the limiting distribution of the sequence of partial sums. The same framework can also be adapted to testing parameter constancy at the marginal distributions: Let \{Z_t\}_{t=1,\ldots,T} \in \mathbb{R}^m denote an i.i.d. sample without assuming a particular parametric model but rather test hypothesis on sample-moments, specified by a function \( g : \mathbb{R}^m \to \mathbb{R}^k \), where \( k \) denotes the number of moment hypotheses that are imposed. A fluctuation test is then based on the partial sums

\[
Q_j = \frac{j}{\sqrt{n}} \sqrt{(S_j - S_n)' \hat{\Omega}^{-1} (S_j - S_n)} \quad \text{with} \quad S_j = \frac{1}{j} \sum_{t=1}^{j} g(Z_t) \tag{A.1}
\]

with covariance matrix \( \Omega \), in practice replaced by some estimator \( \hat{\Omega} \). The limiting process associated with the general fluctuation test (4.1) is closely related to a \( k \)-dimensional Brownian Bridge \( B_k(\pi) \):

\[
Q_j \Rightarrow \sqrt{(\Gamma_k(\pi) - \pi \Gamma_k(1))'(\Gamma_k(\pi) - \pi \Gamma_k(1))} \tag{A.2}
\]

with \( \Gamma(s) \) being a \( k \)-dimensional vector of independent Brownian Motions defined over \( \pi \in [0,1] \). It is now possible to apply different functionals on the limit process, the sup-functional being the most suitable for the alternative of a single regime-change. Consequently

\[
\sup Q_j \to_d \sup ||B_k(\pi)|| \tag{A.3}
\]

So for the CUSUM of squares test, we have \( k = 1 \), if means are also subject to change, \( k = 2 \). Critical values \( c_\alpha \), such that \( P(\sup_{\pi \in \Pi} B_k > c_\alpha) = \alpha \), are tabulated for example in Kiefer [1959] or can easily be simulated. Consequently, \( H_0 \) will be rejected if \( \sup Q_j \) exceeds the quantile of \( \sup ||B_2(\pi)|| \) associated with the desired significance level. The change-point is estimated by

\[
\hat{l}_i = \arg \max_{2 \leq j \leq n-2} Q_j \tag{A.4}
\]

where \( t = 1, n-1, n \) have to be excluded from the sets of potential break points as each sub-sample needs to contain at least two elements. Testing time series with higher observation frequency for structural changes is usually performed under the assumptions of constant means, as for example in Wied et al. [2012a]. The latter authors develop a fluctuation test framework using a CUSUM of squares process, which we adopt for the case of daily financial return series. Since \( \mu \) can not assumed to be constant in low-frequency application, the variance test is slightly generalized in the following by allowing \( \mu_i \) to break simultaneously with \( \sigma_i^2 \). Under the assumption of Gaussian marginal distributions, mean \( \mu \) and variance \( \sigma^2 \) are the only parameters subject to structural changes. It is also possible to embed the \( t_\nu \)-distributional assumption into testing mean-variance stability: if degrees of freedom are assumed to be constant, only one location parameter \( \mu_i \), one scale parameter \( \xi_i \) are subject to change. In a fluctuation test framework, this corresponds to testing constancy of the first
and second moment, \((\mu_{1,i}, \mu_{2,i}, \mu_{3,i}, \mu_{4,i})'\) of \(X_{t,i}\) indexing the dimension under consideration. This follows from the variance definition \(\sigma^2 = \mu_2 - \mu_1^2\) and that \(\mu_1\) is also constant under the null hypothesis. No data transformation is required, such that one can write \(Z_t = X_t\) and it is possible to directly apply a fluctuation test on \((\mu_{1,i}, \mu_{2,i})'\) by assuming that we observe samples from \(Z_{t,i} \sim i.i.d.\) \((\mu_{1,i}, \mu_{2,i}, \mu_{3,i}, \mu_{4,i})\). The moment hypothesis is imposed through

\[
g(Z_t) = (Z_t, Z_t^2)
\]

For bounded third and fourth moments the asymptotic covariance matrix follows from the Central Limit Theorem, applied to the full-sample estimator of \((\mu_{1,i}, \mu_{2,i}, \mu_{3,i}, \mu_{4,i})'\), namely the sample moments \((\hat{\mu}_{1,i}, \hat{\mu}_{2,i}, \hat{\mu}_{3,i}, \hat{\mu}_{4,i})'\):

\[
\sqrt{n} \left( \frac{\hat{\mu}_{1,i}}{\hat{\mu}_{2,i}} \right) \rightarrow_d N \left( \left( \frac{\mu_{1,i}}{\mu_{2,i}} \right); \text{Var} \left( \left( \frac{\hat{\mu}_{1,i}}{\hat{\mu}_{2,i}} \right) \right) \right)
\]

with

\[
\text{Var} \left( \left( \frac{m_{1,i}}{m_{2,i}} \right) \right) = \begin{pmatrix} \text{Var}(Z_t) & \text{Cov}(Z_t, Z_t^2) \\ \text{Cov}(Z_t, Z_t^2) & \text{Var}(Z_t^2) \end{pmatrix} = \begin{pmatrix} \mu_2 - \mu_1^2 & \mu_3 - \mu_1 \mu_2 \\ \mu_3 - \mu_1 \mu_2 & \mu_4 - \mu_2^2 \end{pmatrix}
\]

The fluctuation test statistic \(Q_t\) is computed with

\[
S_j = \left( \frac{1}{j} \sum_{t=1}^j Z_t, \frac{1}{j} \sum_{t=1}^j Z_t^2 \right)'	ext{ where } \hat{\Omega} = \frac{1}{n} \begin{pmatrix} \hat{\mu}_2 - \hat{\mu}_1^2 & \hat{\mu}_3 - \hat{\mu}_1 \hat{\mu}_2 \\ \hat{\mu}_3 - \hat{\mu}_1 \hat{\mu}_2 & \hat{\mu}_4 - \hat{\mu}_2^2 \end{pmatrix}
\]

In the Gaussian case, the relevant moments are obtained as

\[
\begin{align*}
\mu_1 &= \mu \\
\mu_2 &= \mu^2 + \sigma^2 \\
\mu_3 &= \mu^3 + 3\mu \sigma^2 \\
\mu_4 &= \mu^4 + 6\mu^2 \sigma^2 + 3\sigma^4
\end{align*}
\]

such that the asymptotic covariance matrix of the full-sample estimator is given by

\[
\Omega = \frac{1}{n} \begin{pmatrix} \sigma^2 & 2\mu \sigma^2 \\ 2\mu \sigma^2 & 2 \sigma^4 + 4\mu^2 \sigma^2 \end{pmatrix}
\]

Several extensions of practical interest can be tested, one could for example suspect that skewness and/or kurtosis are also subject to structural changes. Simulation evidence however revealed that this additional flexibility does not improve testing in either framework.

### A.2 Testing Constant Cross-Moments

The principle of using partial sums of empirical moments can be extended to testing constant dependency under the assumption that Pearson’s correlation coefficient (or correlation matrix in a higher-dimensional system) is the only parameter of the joint-distribution, that changes between sub-samples and that marginal distribution change only in mean and variance. Trivially satisfied by the multivariate Gaussian, the same methods also apply to a
t-distribution with constant degrees of freedom as the correlation coefficient for standardized data is obtained as

\[ \rho_{12} = \frac{\xi_{12}}{\sqrt{\xi_{11} \cdot \xi_{22}}} = \xi_{12} \sqrt{\xi_{11} \cdot \xi_{22}} = \xi_{12} \]

(A.6)

and so the cross-dispersion \( \xi_{12} \) is the only dependency-shaping parameter subject to change. In order to test for constant correlation, the observed data are now cleaned from possible changes in marginal parameters using the results from the previous section. Therefore we specifically assume that observations are drawn from a latent DGP by

\[
X_t = \begin{pmatrix}
\mu_{1,1} \mathbb{I}_{t \leq \ell_i} + \mu_{1,2} \mathbb{I}_{t > \ell_i} \\
\vdots \\
\mu_{m,1} \mathbb{I}_{t \leq \ell_m} + \mu_{2,m} \mathbb{I}_{t > \ell_m}
\end{pmatrix} + \begin{pmatrix}
\sqrt{\sigma^2_{1,1} \mathbb{I}_{t \leq \ell_i} + \sigma^2_{1,2} \mathbb{I}_{t > \ell_i}} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \sqrt{\sigma^2_{m,1} \mathbb{I}_{t \leq \ell_m} + \sigma^2_{m,2} \mathbb{I}_{t > \ell_m}}
\end{pmatrix} Z_t
\]

Inference on constant marginal distributions is directly based on the observed \( X_t \). Prior to step 2, the data are transformed according to

\[
\tilde{Z}_{i,t} = \frac{X_{i,t} - \hat{\mu}_{i,1} \mathbb{I}_{t \leq \ell_i} - \hat{\mu}_{i,2} \mathbb{I}_{t > \ell_i}}{\sqrt{\hat{\sigma}^2_{i,1} \mathbb{I}_{t \leq \ell_i} + \hat{\sigma}^2_{i,2} \mathbb{I}_{t > \ell_i}}}
\]

if a break is detected or \( \tilde{Z}_{i,t} = \frac{X_{i,t} - \hat{\mu}_{i,1}}{\hat{\sigma}_{i}} \) else (A.7)

for \( i = 1, \ldots, n \). Using \( \tilde{Z}_{i,t} \), partial sums are computed by stacking the elements of the matrix of standardized cross-moments:

\[
\tilde{S}_j = \frac{1}{j} \sum_{t=1}^{j} \left( \tilde{Z}_{1,1} \tilde{Z}_{2,1}, \ldots, \tilde{Z}_{1,1} \tilde{Z}_{m,1}, \tilde{Z}_{2,1} \tilde{Z}_{3,1}, \ldots, \tilde{Z}_{2,1} \tilde{Z}_{m,1}, \ldots, \tilde{Z}_{m-1,1} \tilde{Z}_{m,1} \right)'
\]

(A.8)

Based on the (unobserved) latent DGP \( Z_t \), the test statistic \( \sup \tilde{Q}_j \) would have the limiting distribution \( \sup \| B_{(m-1) \cdot (m-2)} \| \). It remains to find an estimator for the full-sample covariance matrix \( \Omega \). Wied [2015+, forthcoming] suggests a block bootstrap estimator of the corresponding covariance matrix in the case of weakly stationary time series, the transformation prior to step 2 however allows to work under the assumption of strict stationarity, so we employ a simple bootstrap scheme to estimate \( \Omega \) and usual critical values from the Brownian Bridge apply. This way, the effect of stochastic volatility can be absorbed. Bootstrap approximations of the covariance matrix are however no longer valid if breaks are present in the margins. In this case, data are standard piecewise and asymptotic critical values do not apply. Demetrescu and Wied [2018+] therefore suggest to apply a wild bootstrap scheme: Let \( X^*_1, \ldots, X^*_n \) denote a sample from \( X_1, \ldots, X_n \), drawn with replacement. The bootstrap sample is obtained from

\[
X^b_{i,t} = \mu_{i,1} + \frac{(X^*_{i,t} - \mu^*_{i,1})}{\sigma^2_{i,1}^{b}} \sigma_{i,t}
\]

(A.9)

such that in the bootstrap sub-samples at each margin the sample mean \( \mu^b_{i,1} \) and sample variance \( \sigma^2_{i,1}^{b} \) match sample mean \( \mu_{i,1} \) and sample variance \( \sigma^2_{i,1} \) from the original sample. Here \( \mu^*_{i,1} = \frac{1}{n} \sum_{t=1}^{n} X^*_t \) and \( \sigma^2_{i,1}^{*} = \frac{1}{n} \sum_{t=1}^{n} (X^*_t - \mu^*_{i,1})^2 \). In every bootstrap repetition \( b \) the
fluctuation test statistic is computed as in (3.10) and denoted by $\sup Q^b$ and the p-value approximated by

$$\hat{p} = \frac{1}{B} \sum_{b=1}^{B} \mathbb{1}_{\{\sup \hat{Q}^b_j > \sup \hat{Q}_j\}}$$

### A.3 Testing for Constant Copula

Throughout the previous section it has been implicitly assumed that Pearson’s correlation coefficient suffices to describe the dependency in a multivariate system. By imposing this restriction on the dependence structure, many features frequently observed in financial applications are ignored in the testing procedure. Testing for a constant copula is a suitable way to test constant dependency of multivariate random variables beyond the moment hypothesis lined out previously. As before, applying the fluctuation test framework to copulae does not require parametric assumptions but only a test decision regarding the constancy of the marginal distributions. Since the econometric applications of copula-based tests mostly refer to equity data, one may assume constant or zero means over the observation period and use a CUSUM of squares test as presented in Wied et al. [2012a] and standardize the data by

$$\hat{Z}_{i,t} = \frac{X_{i,t}}{\hat{\sigma}_i} \text{ if a break is detected or } \hat{Z}_{i,t} = \frac{X_{i,t}}{\hat{\sigma}_i} \text{ else} \quad (A.10)$$

Based on the piecewise residuals, we follow Bücher et al. [2014] to transform the data onto the 'copula-scale' $[0, 1]^d$, by the empirical distribution function either over the full or the partial samples resulting from a split at $j$:

$$\hat{U}_i(x) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}(X_{t,i} \leq x)$$

$$\hat{U}^{1:j}_i(x) = \frac{1}{j} \sum_{t=1}^{j} \mathbb{1}(X_{t,i} \leq x)$$

$$\hat{U}^{j+1:n}_i(x) = \frac{1}{n-j} \sum_{t=j+1}^{n} \mathbb{1}(X_{t,i} \leq x) \forall x \in \mathbb{R}$$

Define next the full-sample and partial-sample empirical copula by obtained from dividing the sample at a given $j$

$$\hat{C}_n(u) := \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}(\hat{U}_t \leq u)$$

$$\hat{C}^{1:j}_n(u) := \frac{1}{j} \sum_{t=1}^{j} \mathbb{1}(\hat{U}^{1:j}_t \leq u)$$

$$\hat{C}^{j+1:n}_n(u) := \frac{1}{n-j} \sum_{t=j+1}^{n} \mathbb{1}(\hat{U}^{j+1:n}_t \leq u)$$
Then the test is based on the difference process of the partial-sample empirical copulae:

\[ S(j, u) = \frac{j(n - j)}{n^{3/2}} \left( \frac{1}{j} \sum_{t=1}^{j} 1\{\hat{U}_t \leq u\} - \frac{1}{n - j} \sum_{t=j+1}^{n} 1\{\hat{U}_t \leq u\} \right) \]  

(A.11)

Constructing the difference process in this way improves the test statistics of Rémillard [2010] and Bücher and Ruppert [2013] who use the empirical copula over the full sample rather than partial-sample empirical copulae, as pointed out by Bücher et al. [2014] in a simulation study. Thus only the more recent method is used in the subsequent Monte Carlo studies.

The test statistic follows from integrating with respect over \([0, 1]^d\), where in practice a discretization grid has to be chosen and subsequently take the sup-functional over the set of change point candidates \(j \in \{2, \ldots, n-2\}:

\[ T = \sup_j \left( \int_{u[0,1]^d} S(j, u)^2 dC_n(u) \right) \]  

(A.12)

In order to approximate critical or p-values, Bücher et al. [2014] suggest a block multiplier bootstrap scheme that works under strong mixing conditions and is also used in Bücher et al. [2014]\(^1\). Under the assumption of a proper transformation prior to testing for a constant copula, a simplified i.i.d.-multiplier bootstrap can be used in the following way: For each bootstrap repetition \(b\) one draws \(\zeta_{b,t} \stackrel{i.i.d.}{\sim} N(0, 1)\) and computes

\[ \hat{T} = \sup_j \left( \int_{u[0,1]^d} S_b(j, u)^2 dC_n(u) \right) \]

\[ S_b(j, u) = B_b(j, u) - jB_b(1, u) \]

\[ B_b(j, u) = \frac{1}{\sqrt{n}} \sum_{t=1}^{j} \zeta_{b,t} \left( 1\{\hat{U}_t \leq u\} - \hat{C}_n(u) \right) \]  

(A.13)

from where the approximated p-value follows as

\[ \hat{p} = \frac{1}{B} \sum_{b=1}^{B} 1\{\hat{T}_b > T\} \]

**B Bivariate t-Distribution**

In many financial applications with moderate observation frequencies (e.g. monthly or weekly), the heavy-tailed t-distribution yields a better fit than the Gaussian distribution used in section 3.1, see Cont [2001] who collects empirical facts on asset returns. Therefore we now turn to the problem of testing parameter stability under the assumption to observe

\(^1\)The authors also suggest a more advanced bootstrap scheme based on the partial-sample copulae, which however is computationally intractable for the Monte Carlo studies in section 4.
data from
\[(X_{1,t}, X_{2,t}) \overset{i.i.d.}{\sim} t(\mu_{1,1}, \mu_{2,1}, \xi_{11,1}, \xi_{22,1}, \rho_{1, \nu}) \quad \text{for } t = 1, \ldots, l_1\]
\[(X_{1,t}, X_{2,t}) \overset{i.i.d.}{\sim} t(\mu_{1,2}, \mu_{2,2}, \xi_{11,2}, \xi_{22,2}, \rho_{1, \nu}) \quad \text{for } t = l_1 + 1, \ldots, l_2\]
\[(X_{1,t}, X_{2,t}) \overset{i.i.d.}{\sim} t(\mu_{1,2}, \mu_{2,2}, \xi_{11,2}, \xi_{22,2}, \rho_{2, \nu}) \quad \text{for } t = l_2 + 1, \ldots, l_D\]
\[(X_{1,t}, X_{2,t}) \overset{i.i.d.}{\sim} t(\mu_{1,2}, \mu_{2,2}, \xi_{11,2}, \xi_{22,2}, \rho_{2, \nu}) \quad \text{for } t = l_D, \ldots, n\]

where \(\nu\) denotes the degrees of freedom and \(\Xi\) denotes the dispersion matrix, such that the covariance matrix follows as \(\Sigma = \frac{\nu}{\nu - 2} \Xi\). Similar to the Gaussian case, we impose that \(\nu\) and \(\xi\) are bounded away from zero. The correlation coefficient satisfies
\[
\rho_{12} = \frac{\nu \xi_{12}}{\sqrt{\nu - 2} \xi_{11} \sqrt{\nu - 2} \xi_{22}} = \frac{\xi_{12}}{\sqrt{\xi_{11}} \sqrt{\xi_{22}}} 
\]
and the t-distribution is equivalently parametrized - similar to the covariance decomposition in the Gaussian case - in terms of the cross-dispersion and the correlation:
\[
\Xi = \begin{pmatrix}
\xi_{11} & \sqrt{\xi_{11} \xi_{22}} \rho \\
\sqrt{\xi_{11} \xi_{22}} \rho & \xi_{22}
\end{pmatrix}
= \begin{pmatrix}
\xi_1 & \xi_2 \\
\xi_2 & \xi_2
\end{pmatrix}
\]

By the properties of the multivariate t-distribution, each marginal distribution \(i\) satisfies
\[
X_{i,t} \overset{i.i.d.}{\sim} t(\mu_{i,1}, \xi_{i,1}, \nu) \quad \text{for } t = 1, \ldots, l_i
\]
\[
X_{i,t} \overset{i.i.d.}{\sim} t(\mu_{i,2}, \xi_{i,2}, \nu) \quad \text{for } t = l_i + 1, \ldots, n
\]
and can test 1 by setting \(\theta_i = (\mu_i, \xi_i)\). From the distributional assumption, the probability density is given by
\[
f(X_i; \mu, \Xi, \nu) = \frac{\Gamma\left(\frac{\nu + m}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi \nu)^{m/2} |\Xi|^{0.5}} \left(1 + \frac{1}{\nu}(X_i - \mu)'\Xi^{-1}(X_i - \mu)\right)^{-\frac{\nu + m}{2}}
\]
from where the marginal density of dimension \(i\) follows as
\[
f(X_{i,t}; \mu_i, \xi_i, \nu) = \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\nu \xi_i^2}} \left(1 + \frac{(X_{i,t} - \mu_i)^2}{\nu \xi_i^2}\right)^{-\frac{\nu + 1}{2}}
\]
Although degrees of freedom are assumed to be constant in, they nevertheless have to be estimated in finite samples. This is done before testing marginal distributions by maximizing the log-likelihood associated with the joint distribution. No closed-form solution exists for maximizing the log-likelihood, so one has to use numerical methods to find the ML-estimator for \(\mu, \Xi\) and \(\nu\). We refer to Liu and Rubin [1995] for a detailed description of the EMCE-algorithm typically used in this context. Let \(\nu\) denote the ML-estimator of the degrees of freedom for the multivariate distribution, which is now fixed when testing Hypothesis Pair 1 for each margin. Using the same separation of the log-likelihood in terms of \((\mu_1, \xi_1)\) and \((\mu_2, \xi_2)\) as in the Gaussian case, the EMCE-algorithm delivers the corresponding ML-estimator. Full-sample estimators \((\hat{\mu}_0, \hat{\xi}_0)\) are obtained accordingly, which are plugged back into the LR-statistic together with \((\hat{\mu}_1, \hat{\mu}_2, \hat{\xi}_1, \hat{\xi}_2)\). After omitting constants one obtains for
a fixed \( j \)

\[
A_{i,j} = 2\left( (X_i; \hat{\mu}_{i,1}, \hat{\mu}_{i,2}, \hat{\xi}_{i,1}, \hat{\xi}_{i,2}, \bar{\pi}) - L(X_i; \hat{\mu}_{i,0}, \hat{\xi}_{i,0}, \bar{\pi}) \right) \\
= n \cdot \log(\hat{\xi}_{i,0}^2) - j \cdot \log(\hat{\xi}_{i,1}^2) - (n - j) \cdot \log(\hat{\xi}_{i,2}^2) \\
- (\bar{\pi} + 1) \sum_{t=1}^{j} \log \left( 1 + \frac{1}{\bar{\pi}} \left( \frac{X_{i,t} - \hat{\mu}_{i,1}}{\hat{\xi}_{i,1}} \right)^2 \right) \\
- (\bar{\pi} + 1) \sum_{t=j+1}^{n} \log \left( 1 + \frac{1}{\bar{\pi}} \left( \frac{X_{i,t} - \hat{\mu}_{i,2}}{\hat{\xi}_{i,2}} \right)^2 \right) \\
+ (\bar{\pi} + 1) \sum_{t=1}^{n} \log \left( 1 + \frac{1}{\bar{\pi}} \left( \frac{X_{i,t} - \hat{\mu}_{i,0}}{\hat{\xi}_{i,0}} \right)^2 \right)
\]

(B.1)

and Hypothesis Pair 1 is tested using the fact that a reasonable approximation of the distribution of \( \sup_{\delta \in \Theta} A_j \) is given by the distribution of \( \sup_{\pi} B_2(\pi) \).

Testing constant dependency is specified by recognizing that \( \delta_D = \rho_{12} \) under the assumption of constant degrees of freedom. Since the multivariate t-distribution is a location-scale family in \((\mu, \xi)\), a standardization similar to the Gaussian case

\[
\hat{Z}_{i,t} = \frac{X_{i,t} - \hat{\mu}_{i,1}1_{i,t \leq \xi_i} - \hat{\mu}_{i,2}1_{i,t > \xi_i}}{\sqrt{\hat{\xi}_{i,1}1_{i,t \leq \xi_i} + \hat{\xi}_{i,2}1_{i,t > \xi_i}}}
\]

leaves us with

\[
\hat{Z}_t \overset{i.i.d.}{\sim} t(0, 0, 1, 1, \rho, \bar{\pi})
\]

since for standardized data \( \rho = \xi_{12} \). The log-likelihood simplifies considerably, so a simple line search on the first-order condition now suffices to obtain full-sample and partial-sample estimators \( \hat{\xi}_{12} \). The LR-statistic for a constant \( j \) is given by

\[
A_j(\hat{Z}, \hat{\rho}_0, \hat{\rho}_1, \hat{\rho}_2) = n \log(1 - \hat{\rho}_0^2) - j \log(1 - \hat{\rho}_1^2) - (n - j) \log(1 - \hat{\rho}_2^2) \\
+ (\bar{\pi} + 2) \sum_{t=1}^{n} \log \left( 1 + \frac{1}{\bar{\pi}} \left( \frac{\hat{Z}_{i,t}^2 - 2\hat{\rho}_1 \hat{Z}_{i,t} \hat{Z}_{1,t} + \hat{Z}_{2,t}^2}{1 - \hat{\rho}_0^2} \right) \right) \\
- (\bar{\pi} + 2) \sum_{t=1}^{j} \log \left( 1 + \frac{1}{\bar{\pi}} \left( \frac{\hat{Z}_{i,t}^2 - 2\hat{\rho}_1 \hat{Z}_{i,t} \hat{Z}_{1,t} + \hat{Z}_{2,t}^2}{1 - \hat{\rho}_1^2} \right) \right) \\
- (\bar{\pi} + 2) \sum_{t=j+1}^{n} \log \left( 1 + \frac{1}{\bar{\pi}} \left( \frac{\hat{Z}_{i,t}^2 - 2\hat{\rho}_2 \hat{Z}_{i,t} \hat{Z}_{2,t} + \hat{Z}_{2,t}^2}{1 - \hat{\rho}_2^2} \right) \right)
\]

(B.3)

and the test statistic against a single break follows as

\[
\sup_{\delta \in \Theta} \sup_{\pi} A_j
\]

Again, \( \sup_{\pi} B_1(\pi) \) would be a reasonable approximation, if (B.3) were based directly on observed data.

Similar to the bivariate Gaussian distribution examined in section 3.1 from the paper, scenario 1 and 2 are adapted to a t-distribution with degrees of freedom fixed at \( \nu = 5 \) over
the entire sample:

\[(X_{1,t}, X_{2,t}) \overset{i.i.d.}{\sim} t_5(0.05, 0.05, 1, 1, 0.4) \text{ for } t = 1, \ldots, l_1\]

\[(X_{1,t}, X_{2,t}) \overset{i.i.d.}{\sim} t_5(0.06 - 0.01s, 0.05, s_1, 1, 0.4) \text{ for } t = l_1 + 1, \ldots, l_2\]

\[(X_{1,t}, X_{2,t}) \overset{i.i.d.}{\sim} t_5(0.06 - 0.01s; 0.06 - 0.01s, s, s, 0.4) \text{ for } t = l_2 + 1, \ldots, l_D\]

\[(X_{1,t}, X_{2,t}) \overset{i.i.d.}{\sim} t_5(0.06 - 0.01s, 0.06 - 0.01s, s, s, \rho_2) \text{ for } t = l_D, \ldots, n\]

For the timing of the regime-shift, the same values as in the Gaussian case are used.

Table B.1: \(t_5\)-Distribution, Scenario 1: Rejection Rates under \(H_0\)

<table>
<thead>
<tr>
<th>(s)</th>
<th>Fluctuation test</th>
<th>sup-LR test, t, joint</th>
<th>sup-LR test, Gaussian</th>
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<td>asym. boot. (X_1)</td>
<td>asym. boot. (X_1)</td>
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</tr>
<tr>
<td>1</td>
<td>0.018 0.034 0.018</td>
<td>0.708 0.011 0.075</td>
<td>0.244 0.071 0.178</td>
</tr>
<tr>
<td>4/3</td>
<td>0.030 0.060 0.032</td>
<td>0.737 0.008 0.110</td>
<td>0.239 0.074 0.275</td>
</tr>
<tr>
<td>2</td>
<td>0.053 0.087 0.129</td>
<td>0.821 0.028 0.328</td>
<td>0.273 0.079 0.542</td>
</tr>
<tr>
<td>3</td>
<td>0.054 0.090 0.347</td>
<td>0.680 0.031 0.719</td>
<td>0.259 0.079 0.841</td>
</tr>
<tr>
<td>5</td>
<td>0.050 0.074 0.623</td>
<td>0.508 0.017 0.966</td>
<td>0.244 0.075 0.985</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(n = 500)</th>
<th>Fluctuation test</th>
<th>sup-LR test, t, joint</th>
<th>sup-LR test, Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.017 0.051 0.980</td>
<td>0.430 0.009 1</td>
<td>0.332 0.067 1</td>
</tr>
<tr>
<td>1/3</td>
<td>0.014 0.047 0.955</td>
<td>0.436 0.007 1</td>
<td>0.331 0.071 1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.020 0.050 0.729</td>
<td>0.481 0.019 0.953</td>
<td>0.337 0.079 0.975</td>
</tr>
<tr>
<td>0.75</td>
<td>0.032 0.059 0.158</td>
<td>0.811 0.039 0.274</td>
<td>0.367 0.089 0.602</td>
</tr>
<tr>
<td>1</td>
<td>0.026 0.047 0.027</td>
<td>0.672 0.019 0.058</td>
<td>0.357 0.066 0.330</td>
</tr>
<tr>
<td>4/3</td>
<td>0.039 0.062 0.138</td>
<td>0.785 0.033 0.294</td>
<td>0.365 0.084 0.587</td>
</tr>
<tr>
<td>2</td>
<td>0.022 0.046 0.732</td>
<td>0.493 0.023 0.947</td>
<td>0.362 0.078 0.968</td>
</tr>
<tr>
<td>3</td>
<td>0.012 0.038 0.955</td>
<td>0.442 0.006 1</td>
<td>0.340 0.072 1</td>
</tr>
<tr>
<td>5</td>
<td>0.017 0.046 0.984</td>
<td>0.437 0.007 1</td>
<td>0.332 0.067 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(n = 1500)</th>
<th>Fluctuation test</th>
<th>sup-LR test, t, joint</th>
<th>sup-LR test, Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.018 0.059 0.999</td>
<td>0.404 0.010 1</td>
<td>0.399 0.062 1</td>
</tr>
<tr>
<td>1/3</td>
<td>0.017 0.052 0.997</td>
<td>0.402 0.014 1</td>
<td>0.402 0.064 1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.020 0.052 0.975</td>
<td>0.442 0.054 0.982</td>
<td>0.398 0.069 1</td>
</tr>
<tr>
<td>0.75</td>
<td>0.038 0.065 0.420</td>
<td>0.700 0.646 0.706</td>
<td>0.442 0.075 0.867</td>
</tr>
<tr>
<td>1</td>
<td>0.040 0.052 0.035</td>
<td>0.638 0.022 0.056</td>
<td>0.435 0.069 0.406</td>
</tr>
<tr>
<td>4/3</td>
<td>0.037 0.062 0.425</td>
<td>0.722 0.066 0.654</td>
<td>0.435 0.072 0.847</td>
</tr>
<tr>
<td>2</td>
<td>0.019 0.058 0.981</td>
<td>0.436 0.056 0.974</td>
<td>0.404 0.085 1</td>
</tr>
<tr>
<td>3</td>
<td>0.021 0.060 0.994</td>
<td>0.402 0.008 1</td>
<td>0.393 0.062 1</td>
</tr>
<tr>
<td>5</td>
<td>0.022 0.052 0.998</td>
<td>0.406 0.010 1</td>
<td>0.394 0.063 1</td>
</tr>
</tbody>
</table>

The fluctuation test behaves similar to the Gaussian case when testing for constant cross-moments: the nominal level of 5 % is not reached under \(H_0\) when asymptotic critical values are used. As before, the test shows good size properties under the wild bootstrap scheme.
Attention has to be paid in the correctly specified sup-LR test. Although it possesses good power and size properties at the margins in step 1, using asymptotic critical values leads to rejection rates up to 80% under $H_0$. Using the appropriate wild bootstrap scheme puts the empirical rejection rates into acceptable regions, but now constantly falling short of 5% and decreasing towards zero if the margins vary strongly (see the bottom panel of table B.1). Similar to using the correct distributional assumption, testing in the Gaussian framework leads to severe size distortions of the sup-LR test using asymptotic critical values. The test keeps its size under $H_0$ if corrected by the wild bootstrap scheme and looks preferable in terms of size to the (computationally more intensive) sup-LR test under the correct distributional specification.

Figure B.1: $t_5$-Distribution, n=100, Scenario 2: Empirical Power

![Figure B.1](image1.png)

Figure B.2: $t_5$-Distribution, n=500, Scenario 2: Empirical Power

![Figure B.2](image2.png)
Since sample sizes of 500 are hardly reached for monthly or quarterly data, we suggest using more elaborate methods relying on the t-distribution should only be used, if the sample size is sufficiently large.

Findings on power draw a picture similar to the Gaussian case: there is some inconclusiveness in small samples, but figure B.1 - B.3 show the sup-LR test gaining power faster than the fluctuation test, even though it is found to be conservative. We defer the discussion of dimensionality effects in this parametric class to the next section on copulae and directly move on to the accuracy of break-point estimators. Table B.2 again shows superiority of the sup-LR tests over non-parametric methods in terms of bias and root mean-squared error of the break-point estimator for $1_D$ and $l_1$ in larger samples. We further conclude that the more elaborate methods relying on the t-distribution should only be used, if the sample size is sufficiently large.

Table B.2: $t_5$-Distribution, Scenario 2: Break Point Estimation

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Fluctuation test</th>
<th>sup-LR test</th>
<th>sup-LR test, Gaussian Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{bias}(\rho)$</td>
<td>$\text{rmse}(\rho)$</td>
<td>$\text{bias}(\rho)$</td>
</tr>
<tr>
<td>$n=500$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>-10.12</td>
<td>30.04</td>
<td>10.12</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>-20.24</td>
<td>40.44</td>
<td>20.24</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>-30.35</td>
<td>50.85</td>
<td>30.35</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>-40.49</td>
<td>61.29</td>
<td>40.49</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>-50.63</td>
<td>71.73</td>
<td>50.63</td>
</tr>
</tbody>
</table>

Figure B.3: $t_5$-Distribution, n=1500, Scenario 2: Empirical Power
the misspecified Gaussian sup-LR test in a shift contagion scenario and employ the respective bootstrap method. As in the bivariate Gaussian case it could prove useful to additionally apply the fluctuation test, if a reduction in correlation is suspected. There also may be situations with more than two dimensions and a sample size too small to obtain reliable parameter estimates under the t-distribution specification, for example \( m = 3 \) and \( n = 200 \). Such cases are not formally considered here and are left for future research, based on the findings on the multivariate Gaussian one can suspect that - using an appropriate bootstrap scheme - the sup-LR test is preferable here. Should the sample be large enough to permit reliable estimation, the preceding findings favour the sup-LR test using the parametric approach. Extensions to the multivariate case are obtained analogously to the Gaussian case. Because of the high computational effort, the lack of additional insight and the more flexible way to handle t-distributed random variables using copulae, this is not pursued further.

C  Sup-LR Test for Gaussian Copula

As alternative to using t-Copula as in section 3.2 in the main paper, step 2 can also be based on the copula associated with the Gaussian distribution assumption. Step 1 remains unchanged, however the data are now (piecewise) transformed onto the copula scale by

\[
\hat{U}_{i,t} = F(X_i, \hat{\mu}_{i1}, \hat{\sigma}_{i1}) \quad \text{for } t = 1, \ldots, \hat{l}_i \\
\hat{U}_{i,t} = F(X_i, \hat{\mu}_{i2}, \hat{\sigma}_{i2}) \quad \text{for } t = \hat{l}_i + 1, \ldots, n \quad \text{if the test rejects} \\
\hat{U}_{i,t} = F(X_i, \hat{\mu}_{i0}, \hat{\sigma}_{i0}) \quad \text{for } t = 1, \ldots, n \quad \text{if not}
\]  

(C.1)

The pseudo-observations are then used to estimate the dependency parameter (i.e. the correlation matrix) of the Gaussian copula under the null and alternative hypothesis. Consider next the density of the Gaussian copula

\[
f(\hat{U}; P) = |R|^{-\frac{1}{2}} \exp \left( -\frac{1}{2} \hat{U}'(R^{-1} - I)\hat{U} \right)
\]

from where the full-sample log-likelihood

\[
L(\hat{U}; P_0) = -\frac{n}{2} |R| - \frac{1}{2} \sum_{t=1}^{n} \hat{U}'_t (R_0^{-1} - I)\hat{U}_t
\]

and the partial-sample log-likelihood

\[
L(\hat{U}; P_1, P_2) = -\frac{j}{2} |R_1| - \frac{n-j}{2} |R_2| - \frac{1}{2} \sum_{t=1}^{j} \hat{U}'_t (R_1^{-1} - I)\hat{U}_t - \frac{1}{2} \sum_{t=j+1}^{n} \hat{U}'_t (R_2^{-1} - I)\hat{U}_t
\]

are obtained. Let \( \hat{R}_0, \hat{R}_1 \) and \( \hat{R}_2 \) denote the ML-estimators for the correlation matrix of the full sample and each sub-sample. Evaluating the log-likelihood at the respective parameter estimates gives the test statistic for a fixed \( j \) as

\[
A_j = 2(L(\hat{U}; \hat{R}_1, \hat{R}_2) - L(\hat{U}; \hat{R}_0)).
\]  

(C.2)
Had one based the test statistic on the unobserved $Z_t$, a reasonable approximation of the critical value associated with the sup-functional $\sup_{\pi n \leq j \leq \pi n} \max_{\pi \in \mathbb{P}} B_j(m-1)m/2(\pi)$. 

D Robustness of Copula-Based Methods of GARCH-residuals

This section provides evidence on robustness of the Copula-methods based on GARCH-residuals in the sup-LR framework used in the application to financial sector stocks. For the corresponding simulation evidence on tests based on empirical copulae, we refer to B"ucher and Ruppert [2013]. Specifically we generate data according in similar fashion to section 3.2

\[
\begin{align*}
U_t & \sim \text{i.i.d.} \quad C_t(P_1,4) \quad \text{for } t = 1, \ldots, l_1 \\
U_t & \sim \text{i.i.d.} \quad C_t(P_2,4) \quad \text{for } t = l_1, \ldots, n \\
Z_{1,t} &= \Phi^{-1}(U_{1,t}) \quad Z_{2,t} = \Phi^{-1}(U_{2,t}) \\
\sigma_{1,t}^2 &= \alpha Z_{1,t}^2 + \beta \sigma_{1,t-1}^2 \\
\sigma_{2,t}^2 &= \alpha Z_{2,t}^2 + \beta \sigma_{2,t-1}^2 \\
X_{1,t} &= \sigma_{1,t} Z_{1,t} \quad \text{for } t = 1, \ldots, l_1 \\
X_{1,t} &= s \sigma_{1,t} Z_{1,t} \quad \text{for } t = l_1, \ldots, n \\
X_{2,t} &= \sigma_{2,t} Z_{2,t} \quad \text{for } t = 1, \ldots, l_2 \\
X_{2,t} &= s \sigma_{2,t} Z_{2,t} \quad \text{for } t = l_2, \ldots, n \\
\end{align*}
\]

with $\sigma_{1,1}^2 = \sigma_{2,1}^2 = 1$ and

\[
\begin{align*}
P_1 &= \begin{pmatrix} 1 & 0.4 \\ 0.4 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{pmatrix}
\end{align*}
\]
Table D.1: t-Copula, Scenario 1B: Rejection Rates under $H_0$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$n = 500$</th>
<th>Copula $X_1$</th>
<th>Copula $X_2$</th>
<th>$n = 1500$</th>
<th>Copula $X_1$</th>
<th>Copula $X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.088</td>
<td>0.529</td>
<td>1</td>
<td>0.047</td>
<td>0.91</td>
<td>1</td>
</tr>
<tr>
<td>1/3</td>
<td>0.143</td>
<td>0.078</td>
<td>0.343</td>
<td>0.054</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.125</td>
<td>0.029</td>
<td>0.096</td>
<td>0.061</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.75</td>
<td>0.085</td>
<td>0.044</td>
<td>0.063</td>
<td>0.073</td>
<td>0.998</td>
<td>0.995</td>
</tr>
<tr>
<td>1</td>
<td>0.050</td>
<td>0.053</td>
<td>0.040</td>
<td>0.060</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4/3</td>
<td>0.062</td>
<td>0.594</td>
<td>0.488</td>
<td>0.049</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.062</td>
<td>1</td>
<td>0.992</td>
<td>0.049</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0.051</td>
<td>1</td>
<td>1</td>
<td>0.045</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0.055</td>
<td>1</td>
<td>1</td>
<td>0.044</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

E Application to European Financial Sector Stocks

In a second practical example, we apply the non-parametric methods testing constant copula as lined out in section 3.2 to EURO STOXX 50 financial sector data around the financial crisis following the Lehman Brothers insolvency. Therefore we take daily log-returns of BNP Paribas, Santander, Allianz, AXA, ING Group, BBVA, Intesa Sanpaolo, Société Générale, Munich Re and Deutsche Bank from April-01-2004 to April-01-2010. In applications of this kind, involving a higher observation frequency, the (piecewise) i.i.d. has to be relaxed in favour of a piecewise weak stationarity assumption, since volatility clustering in the marginal return distributions is frequently observed in the market (some empirical evidence on conditional heteroskedasticity in asset returns are also found in Cont [2001]). Often this phenomenon is modeled using GARCH-type models, Chuang and Lee [2014] and Bartram et al. [2007] use for example a GJR-GARCH(1,1)-t model while Candelon and Manner [2010] employ a SWARCH-approach and Andreou and Ghysels [2003] use GARCH(1,1)-t and GARCH(1,1)-normal models. In order to focus on the relevant aspects in Sequential Procedures, we restrict ourselves to the simplest case of an univariate GARCH(1,1)-model from Bollerslev [1986]:

\[
X_{i,t} = \mu_i + \sqrt{h_{i,t}} Z_{i,t}, \quad h_{i,1} = c_{i,1} + \alpha_{i,1} X_{i,t-1}^2 + \beta_{i,1} h_{i,1}, \quad Z_{i,t} \overset{i.i.d.}{\sim} (0, 1) \quad \text{for } t = 1, \ldots, l_i
\]

\[
X_{i,t} = \mu_i + \sqrt{h_{i,2}} Z_{i,t}, \quad h_{i,2} = c_{i,2} + \alpha_{i,2} X_{i,t-1}^2 + \beta_{i,1} h_{i,1}, \quad Z_{i,t} \overset{i.i.d.}{\sim} (0, 1) \quad \text{for } t = l_i, \ldots, n
\]

for $i = 1, \ldots, m$  

(E.1)
We assume constancy of $\mu_i$ over the entire sample, since in applications with financial crises or shift contagion (with at most one break) one often has samples of two or three years. Over such a time horizon the zero mean assumption can be maintained. This is in contrast with the methods useful in low-frequency observations lined out earlier, where the expected returns may change visibly. In order to test for stability of the GARCH-process one could use a full or partial sup-LR testing procedure by holding some of the parameters constant, for example $\alpha_i$ and $\beta_i$. However the purpose of using a (piecewise) GARCH-model at each margin is to eliminate heteroskedasticity from the data prior to step 2 in the sequential procedure. Since conditional heteroskedasticity is covered by the assumptions in e.g. Wied et al. [2012a] for the fluctuation test framework and Qu and Perron [2007] for the sup-LR test framework (for the technical aspects, see Carrasco and Chen [2002]), it suffices here to test for constancy of the unconditional variance $\bar{\sigma}_i^2 = c_i - \alpha_i - \beta_i$ as before and the problem boils down to incorporate the conditional heteroskedasticity into the covariance matrix $\Omega$. Blatt et al. [2015] suggest to use the commonly used procedure of Newey and West [1984] and obtain critical values by simulation. A heteroskedasticity-consistent covariance matrix estimator is proposed in Wied et al. [2012a], another would be using the kernel-based method by Andrews [1991].

In order to describe the transformation prior to step 2, let $(\hat{c}_{i,0}, \hat{\alpha}_{i,0}, \hat{\beta}_{i,0})$ denote the full-sample estimators for the GARCH-model and $(\hat{c}_{i,1}, \hat{\alpha}_{i,1}, \hat{\beta}_{i,1}, \hat{c}_{i,2}, \hat{\alpha}_{i,2}, \hat{\beta}_{i,2})$ the estimators over the each partial sample. Conditional on the test results either transform the original data by

$$
\hat{h}_{i,t} = \hat{c}_{i,0} + \hat{\alpha}_{i,0} X_{i,t-1} + \hat{\beta}_{i,0} h_{i,t-1} \\
\hat{Z}_{i,t} = \frac{X_{i,t} - \mu_{i,0}}{\sqrt{\hat{h}_{i,t}}} 
$$

if no break was detected or by

$$
\hat{h}_{i,t} = \hat{c}_{i,1} + \hat{\alpha}_{i,1} X_{i,t-1} + \hat{\beta}_{i,1} h_{i,t-1} \\
\hat{Z}_{i,t} = \frac{X_{i,t} - \mu_{i,1}}{\sqrt{\hat{h}_{i,t}}} \quad \text{for } t = 1, \ldots, l_i \\
\hat{h}_{i,t} = \hat{c}_{i,2} + \hat{\alpha}_{i,2} X_{i,t-1} + \hat{\beta}_{i,2} h_{i,t-1} \\
\hat{Z}_{i,t} = \frac{X_{i,t} - \mu_{i,2}}{\sqrt{\hat{h}_{i,t}}} \quad \text{for } t = l_i, \ldots, n
$$

(E.2)

Having obtained GARCH-residuals, we follow Chan et al. [2009] and use the residual-copula to test constancy of correlation between the innovations $\epsilon_{i,t}$. Based on critical values 12.35 for the demeaned sup-LR test on constant unconditional variances and 1.628 for the CUSUM of squares test (at 99 % nominal level respectively), $H_0$ (constant variances) is rejected at each margin. Although there are some minor differences in the break point estimates, this can also be seen graphically from rolling volatilities (annualized standard deviations) in appendix E. Here, change-point and partial-sample estimators for $\sigma_{i,1}$ and $\sigma_{i,2}$ are based on the results of the sup-LR test.
Table E.1: Estimation of EURO STOXX Financial Sector Stocks

<table>
<thead>
<tr>
<th></th>
<th>Empirical copula test</th>
<th>sup-LR test, t-Copula</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Q_i$  $\hat{\lambda}_i$ $\hat{\sigma}_1$ $\hat{\sigma}_2$</td>
<td>$A_i$  $\hat{\lambda}_i$ $\hat{\sigma}_1$ $\hat{\sigma}_2$</td>
</tr>
<tr>
<td>BNP Paribas</td>
<td>4.92  2008-01-18 20.58 59.09</td>
<td>825.17  2008-01-17 20.48 59.10</td>
</tr>
<tr>
<td>Santander</td>
<td>5.41  2008-01-15 17.59 48.17</td>
<td>759.58  2008-01-14 17.48 48.20</td>
</tr>
<tr>
<td>Allianz</td>
<td>4.17  2008-01-18 20.69 54.13</td>
<td>703.44  2008-01-17 20.48 54.22</td>
</tr>
<tr>
<td>AXA</td>
<td>5.22  2008-06-26 25.45 69.38</td>
<td>833.64  2008-01-17 22.40 65.03</td>
</tr>
<tr>
<td>ING Group</td>
<td>5.47  2008-09-15 23.76 98.88</td>
<td>1579.87  2008-01-17 18.62 85.97</td>
</tr>
<tr>
<td>BBVA</td>
<td>5.51  2008-01-15 17.11 44.92</td>
<td>698.93  2008-01-14 17.03 44.94</td>
</tr>
<tr>
<td>Intesa Sanpaolo</td>
<td>4.83  2008-07-11 20.13 57.15</td>
<td>805.61  2008-07-10 20.00 57.20</td>
</tr>
<tr>
<td>Munich Re</td>
<td>4.12  2008-01-15 17.80 37.74</td>
<td>431.71  2008-01-14 17.76 37.76</td>
</tr>
<tr>
<td>Deutsche Bank</td>
<td>5.54  2008-07-01 22.06 72.15</td>
<td>1028.61  2008-06-24 21.86 71.94</td>
</tr>
<tr>
<td>Copula</td>
<td>0.0025 2007-06-05</td>
<td>294.99  2008-01-15 $\nu = 16.9$</td>
</tr>
</tbody>
</table>

As for the correlation matrix $P$, the critical value associated with $\sup_{\mathcal{B}_{45}(\pi)}$ is approximately 85.00 at 99 %-level and 94.82 at 99.9 %, so the null hypothesis of parameter stability is overwhelmingly rejected. From the simulation results on $\hat{I}_D$ in section 4.4, 2008-01-15 might be the more reliable change-point estimate, largely coinciding with variance change-points of the single stocks. In addition, the change-point estimate delivered by the nonparametric test is somewhat peculiar, since it falls into a period before the actual extent of the financial turmoil became publicly known. Using a sequential procedure again avoids biased change-point estimates, that could well occur using non-sequential procedures. Since some (but not all) breakpoints coincide, it may be a promising approach to apply the change-point distinctiveness test from Perron and Oka [2011].
Figure E.1: EURO STOXX Financial Sector Stocks, Rolling Volatilities
F Application to Commodity and Equity Index Data

A third empirical application uses the methods subject to the simulation studies section 3.1 and appendix B: Daily log-returns of real estate and equity indices are sequentially tested for constancy of correlation once under the Gaussian (section 3.1 in the main paper) and once under the assumption of a bivariate t-distribution (appendix B). Both tests are benchmarked against the non-parametric fluctuation test outlined in Appendix A. We test for Crude Oil spot market returns and the European equity sector, which we proxy by the EUROSTOXX50 over the time-period 1991-04-17 to 2003-03-05. Foreign involvement in petrol-exporting countries has been fairly low following the early 1990s until 2003. Additionally events in the late 1980s and later technological changes in oil production, the financial crises and relaxed monetary policy probably did not influence the fundamental market environment over the sample. However markets experienced a period of increased volatility around 2000, associated with events such as the burst of the dotcom-bubble among others. This can be observed in figure F.3 and F.4. Rolling correlations in figure F.2 however indicate a rather stable correlation pattern over the test period and thus making the sample a plausible candidate to test for Hypothesis Pair 3. Reported numbers are annualized (business-)daily log-returns and their volatilities (annualized standard-deviations) in percent.

Figure F.1: Estimation of European Crude Oil and Equity Data

<table>
<thead>
<tr>
<th></th>
<th>Fluctuation Test</th>
<th>sup-LR test, Gauss</th>
<th>sup-LR test, t</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Crude Oil</td>
<td>Equity</td>
<td>Crude Oil</td>
</tr>
<tr>
<td>test statistic</td>
<td>4.78</td>
<td>7.76</td>
<td>378.32</td>
</tr>
<tr>
<td>$\hat{\mu}_1$</td>
<td>-4.32</td>
<td>15.00</td>
<td>-0.57</td>
</tr>
<tr>
<td>$\hat{\mu}_2$</td>
<td>16.27</td>
<td>-3.94</td>
<td>8.12</td>
</tr>
<tr>
<td>$\hat{\sigma}_1$</td>
<td>26.22</td>
<td>12.36</td>
<td>23.59</td>
</tr>
<tr>
<td>$\hat{\sigma}_2$</td>
<td>42.81</td>
<td>27.56</td>
<td>40.28</td>
</tr>
<tr>
<td>test statistic</td>
<td>0.771</td>
<td>4.9</td>
<td>7.82</td>
</tr>
<tr>
<td>p-value (boot)</td>
<td>0.595</td>
<td>0.495</td>
<td>0.745</td>
</tr>
<tr>
<td>$\rho_0$</td>
<td>0.0141</td>
<td>0.008</td>
<td>0.032</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>-0.0433</td>
<td>0.011</td>
<td>0.027</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>0.0438</td>
<td>-0.002</td>
<td>0.022</td>
</tr>
<tr>
<td>$\hat{\nu}$</td>
<td></td>
<td></td>
<td>4.99</td>
</tr>
</tbody>
</table>

All procedures strongly reject the hypothesis of constant margins, the critical values at 99 % for the fluctuation test being 1.84 and for the sup-LR tests 15.51. Break-point estimates lie together very closely for both specifications of the sup-LR tests; based on results in table B.2 we favor the estimates based on the sup-LR test with t-distributional assumption. When it comes to testing constant correlation, our empirical findings from section 3.1 and appendix B directly carry over to this particular example: Following table B.1, it is crucial to apply a suitable bootstrap here. Using the bootstrapped p-values around or larger than

\footnote{Europe Brent, Data is taken from the U.S. Energy Information Administration: https://www.eia.gov/dnav/pet/hist/}

\footnote{ISIN: EU0009658145, returns are calculated from the closing price of the last trading day each month.}
0.5, neither fluctuation test and not the sup-LR tests reject Hypothesis Pair 3. Had one used the incorrect asymptotic value for the sup-LR test, which is 7.17 at 90% confidence level, one might incorrectly reject $H_0$ using the empirically plausible t-distributional assumption. It has been previously established that incorrectly assuming constant variances when testing for constant correlation - implicitly by considering covariances as Aue et al. [2009] or explicitly by directly using the procedure of Wied et al. [2012b] - leads to flawed inference. But, as the preceding application points out, even if changes at the marginal distributions are taken into account correctly, applying invalid standard asymptotics may lead to incorrectly rejecting constant cross-sectional dependence.

Figure F.2: Rolling Correlations, Equity and Crude Oil
References


