Abstract

We propose a new non-parametric test for detecting relevant breaks in copula functions. We assume that the data is driven by two non-equal copulas $C_1$ and $C_2$. Under the null hypothesis, the copula difference within an appropriate norm is smaller than a certain positive adjustable threshold $\Delta$. Within the alternative hypothesis, the copula difference exceeds the fixed value $\Delta$. The test is based on a cumulative sum approach of the empirical copula with sequentially estimated marginals. We propose a bootstrap procedure to compute critical values. The Monte Carlo simulation indicates that the test results in a reasonable sized and powered testing procedure. A real data application of the DAX30 up to cross sectional dimension $N = 30$ shows the test’ ability to detect relevant break points.

Keywords: Relevant change, Copula, Break testing, Bootstrap, CUSUM

JEL codes: C12, C13, C32
1. INTRODUCTION

It is well known that dependencies within a portfolio increase in times of financial crisis (cf. Aloui, Aissa, and Nguyen (2011)). From a portfolio manager point of view the increase of the dependencies is disadvantageous. In fact, investors are interested in decreasing the dependencies by rescheduling the portfolio to lower the risk of losses, which is known as the diversification effect. One of many approaches to detect those changes in the dependence structure is to test for changes in the copula function. For instance, Busetti and Harvey (2011), Brodsky, Penikas, Safaryan et al. (2009) and Krämer and van Kampen (2011) have designed nonparametric tests for breaks in the copula in a fixed point considering $N$-dimensional random vectors. Bücher and Ruppert (2013) extended their approaches by testing for overall constancy of the copula in the case of known marginal distributions, while the test of constancy suggested in Bücher, Kojadinovic, Rohmer, and Segers (2014) considers sequentially estimated marginals. Wied, Dehling, van Kampen, and Vogel (2013) propose a test for changes in Spearman’s rho, Dehling, Vogel, Wendler, and Wied (2016) consider a test for changes in Kendall’s tau. Manner, Stark, and Wied (2018+) construct a parametric test for detecting breaks in the parameters of factor copula models. The above mentioned tests can be applied to detect and quantify contagions between different financial markets or to construct optimal portfolios.

The above proposed methods test for the "classical" hypothesis, meaning that they test for stationarity in a sequence of random vectors $\{X_j\}_{j=1}^T$ with $X_j \in \mathbb{R}^N$, i.e.

$$H_0 : X_1, X_2, \ldots, X_T \sim F.$$ 

with the alternative in the simplest case of one structural breakpoint in time (cf. Dette and Wied (2016))

$$H_1 : X_1, X_2, \ldots, X_j \sim F_1 \text{ and } X_{j+1}, \ldots, X_T \sim F_2,$$

where the distribution function changes from $F_1$ to $F_2$ with $F_1 \overset{d}{=} F_2$ at time $j \in \{1, \ldots, T\}$. A general issue of such hypothesis testing is the consistency problem, i.e. any consistent test will detect any arbitrary small change in the parameters if the sample size is sufficiently large. This discrepancy was mentioned for the first time in 1938 by Berkson (1938).

Beyond that, in the case of small changes the rejection of the null might result in an unnecessary break point estimation and an expensive adjustment of the considered model. In practice, small changes in the data might not be crucial, since they do not necessarily add up to significant changes. Thus, the gain derived by the detected break point could be negatively overcompensated by the costs of adjusting the model (e.g., in case of portfolio theory these can be interpreted as transaction costs) or to be short and to the point: Significance does not necessarily imply relevance.

Thus, we impose the more realistic assumption that our sequence of random vectors $\{X_j\}_{j=1}^T$ with $X_j \in \mathbb{R}^N$ is driven by the distribution function $F_1$ and $F_2$, i.e. $X_1, X_2, \ldots, X_j \sim \ldots$
Let $X_{j+1}, \ldots, X_T \sim F_2$, for some $j \in \{1, \ldots, T\}$ such that
\[ H_0 : \|F_1 - F_2\| \leq \Delta \quad \text{versus} \quad H_1 : \|F_1 - F_2\| > \Delta, \tag{1} \]
where $\|\cdot\|$ is an appropriate norm and $\Delta > 0$ a fixed adjustable size. The framework in (1) allows for a break in the data (classical break point tests do not) and the adjustable size $\Delta$ could serve as a measure to control for the extent of the change.

Dette and Wied (2016) have proposed a general approach to this problem. Later on, Dette, Wu, and Zhou (2018) and Dette and Gösmann (2017) have extended this to the detection of changes in second-order characteristics and to high-dimensional models, respectively. Motivated by their analysis we are interested in augmenting the literature of testing for relevant breaks in the copula of random vectors by a nonparametric testing procedure that detects relevant changes in the copula function with sequentially estimated marginal distributions. Thus, the testing problem is given by
\[ H_0 : \|C_1 - C_2\| \leq \Delta \quad \text{versus} \quad H_1 : \|C_1 - C_2\| > \Delta, \]
where $C_1 \neq C_2$ are copulas and $\Delta > 0$ fixed.

Coming back to portfolio management, a small increase in the dependence structure of a portfolio does not necessarily indicate the need to reschedule the portfolio, since transaction costs could overcompensate the benefits of the new, more risk diversified portfolio. Only a relevant change in the dependence structure should result in rescheduling the portfolio.

In our empirical application we analyzed the German DAX30 data of cross sectional dimension $N = 30$ between October 2003 and July 2015. Here, $\Delta$ could be interpreted as the largest admissible copula difference such that the relevant change hypothesis is not rejected. Every other choice of $\Delta$ that is smaller leads to a rejection of the null hypothesis. As a result, $\Delta$ can also be considered as a measure, that quantifies the extent of a crisis.

The rest of the paper is structured as follows. Section 2 introduces the considered null hypothesis and test statistic, where Section 3 presents the bootstrap procedure to determine critical values to perform the test. Results from the Monte Carlo simulations can be found in Section 4. Section 5 presents our empirical application and Section 6 concludes the paper. A supplemental Appendix provides theoretical background.

### 2. RELEVANT CHANGE AND TEST STATISTIC

In this section we introduce the null hypothesis, the assumptions and the relevant change characteristic of our testing procedure in a fully non-parametric setting. Let $X_1, \ldots, X_T$ denote $N$-dimensional random vectors and $U_1, \ldots, U_T$ the vector of the marginal distributions, i.e. $U_t := (F_1(X_{t1}), \ldots, F_N(X_{tN}))$ for $t = 1, \ldots, T$ where $F_i(\cdot)$ is the $i$-th marginal such that
\[
U_1, \ldots, U_{sT} \sim C_1(u) \quad U_{sT+1}, \ldots, U_T \sim C_2(u), \tag{2}
\]
where \( u \in [0, 1]^N \) and \( C_1, C_2 : [0, 1]^N \rightarrow [0, 1] \) are copulas which capture the dependencies between the components of \( X_1, \ldots, X_{\lfloor sT \rfloor} \) and \( X_{\lfloor sT \rfloor + 1}, \ldots, X_T \), respectively. Here, \( \lfloor sT \rfloor \) denotes the change point in time, where \( T \) is the size of the sample and \( s \in (0, 1) \). Note, that the model set-up (2) is valid under both the null and the alternative hypothesis. In order to achieve reliable results, classical concepts of dependencies (e.g. \( U_1, \ldots, U_T \) is stationary and strong mixing with coefficients \( \alpha_n \) converging sufficiently fast to 0) are not applicable any more in the setting of detecting relevant changepoints, because the general model set-up (2) of relevant changepoint analysis allows the sequence \( U_1, \ldots, U_T \) to be non stationary. That is why we have to impose the assumption of a triangular array, that is \( \alpha \)-mixing\(^1\).

To aggregate over \( u \), we consider the \( L^2 \)-norm \( \| \cdot \|_{L^2} \). Thus, the null hypothesis of no relevant change in the copula function is given by

\[
H_0 : \| C_1(u) - C_2(u) \|_{L^2} \leq \Delta
\]

versus the alternative

\[
H_1 : \| C_1(u) - C_2(u) \|_{L^2} > \Delta,
\]

where \( \| \cdot \|_{L^2} \) is the \( L^2 \)-norm and \( \Delta > 0 \) fixed. For every \( u \in [0, 1]^N \) and \( t \in (0, 1) \) the cumulative sum (CUSUM) type process for detecting changes in the copula is then

\[
\hat{U}_T^*(t, u) := t(1-t) \left( \frac{1}{[tT]} \sum_{i=1}^{[tT]} Z_i^{1:[tT]}(u) - \frac{1}{T-[tT]} \sum_{i=[tT]+1}^{T} Z_i^{[tT]+1:T}(u) \right), \tag{3}
\]

where \( Z_i^{t_1:t_2}(u) := \mathbb{1}\{\hat{F}_{t_1:t_2}^i(X_{i1}) \leq u_1, \ldots, \hat{F}_{t_1:t_2}^i(X_{iN}) \leq u_N\}, t_1 < t_2 \in \{1, \ldots, T\} \) for \( i = 1, \ldots, T \) and \( \mathbb{1}\{\cdot\} \) the indicator function. Here \( \hat{F}_{t_1:t_2}^j(\cdot) \) is the empirical distribution function, using data information between \( t_1 \) and \( t_2 \) and is defined as

\[
\hat{F}_{t_1:t_2}^{j}(x) := \frac{1}{t_2 - t_1 + 1} \sum_{i=t_1}^{t_2} \mathbb{1}\{X_{ij} \leq x\}, \quad j = 1, \ldots, N.
\]

For the derivation of our testing procedure we now consider \( \hat{U}_T(t, u) \) defined as

\[
\hat{U}_T(t, u) := t(1-t) \left( \frac{1}{[tT]} \sum_{i=1}^{[tT]} Z_i(u) - \frac{1}{T-[tT]} \sum_{i=[tT]+1}^{T} Z_i(u) \right). \tag{4}
\]

\(^1\)Due to the fact that this discussion is very technical we shifted the details to the Supplemental Appendix
where \( Z_i(u) := 1 \{ F_1(X_{i1}) \leq u_1, \ldots, F_N(X_{iN}) \leq u_N \} \), \( t_1 < t_2 \in \{ 1, \ldots, T \} \) with \( F_i \) as known marginals, \( i = 1, \ldots, T \). For fixed \( s \in (0, 1) \), a straightforward calculation yields

\[
\lim_{T \to \infty} \mathbb{E}[\hat{U}_T(t, u)] = \begin{cases} 
  s(1-t)(C_1(u) - C_2(u)), & s \leq t \\
  t(1-s)(C_1(u) - C_2(u)), & s > t,
\end{cases}
\]  

(5)

where we have to distinguish between data before and after the breakpoint \([sT]\). In the next step, we want to lose the quantile and time dimension \( u \) and \( t \), respectively. For this purpose, we consider the \( L^2 \)-norm and obtain

\[
L(t) := \lim_{T \to \infty} \mathbb{E}[\|\hat{U}_T(t, u)\|_{L^2}^2] = \begin{cases} 
  s^2(1-t)^2\|C_1(u) - C_2(u)\|^2_{L^2}, & t > s \\
  (1-s)^2t^2\|C_1(u) - C_2(u)\|^2_{L^2}, & t \leq s,
\end{cases}
\]

for every norm of the type \( \|f(\cdot, u)\|^2_{L^2} := \int_{[0,1]^N} f(\cdot, u)^2 du \). Integrating out \( t \) yields

\[
\int_0^1 L(t)dt = \frac{s^2(1-s)^2}{3}\|C_1(u) - C_2(u)\|^2_{L^2}.
\]  

(6)

Thus, integrating out \( t \) from the empirical counterpart \( \hat{L}_T(t) := \|\hat{U}_T(t, u)\|_{L^2}^2 \) yields the test statistic \( \hat{\kappa}_T \) for the initial problem of detecting the relevant change

\[
\hat{\kappa}_T := \int_0^1 \hat{L}_T(t)dt.
\]  

(7)

We use \( \hat{s} := \text{argmax} \|\hat{U}_T(s, u)\|_{L^2} \) as the natural argmax estimator for the changepoint location fraction \( s^3 \). We reject the null hypothesis of no relevant change if the test statistic (7) less the adjusted centring \( \frac{s^2(1-s)^2}{3}\|C_1(u) - C_2(u)\|^2_{L^2} \) deviates too far from zero. If the marginal distributions are known the limiting distribution of the process

\[
\sqrt{T} \left( \int_0^1 \hat{L}_T(t)dt - \frac{s^2(1-s)^2}{3}\|C_1(u) - C_2(u)\|^2_{L^2} \right),
\]

(8)

is normal which is shown in the Supplemental Appendix.

Due to the high computational effort in high dimensions using the \( L^2 \)-norm it could be reasonable to only test for specific points \( q \) in the copula, e.g. \( q \) could be chosen as the value that maximizes the copula difference, i.e. \( q := \sup_{u \in [0,1]^N} |C_1(u) - C_2(u)| \). For this purpose

\footnote{For the very detailed derivation of the testing procedure we refer to the Supplemental Appendix.}

\footnote{Note, \( \hat{s} \) is a superconsistent estimator of the changepoint fraction \( s \) with convergence rate \( T \) (cf. Dette and Wied (2016)).}
we fix \( q = (q_1, \ldots, q_N)' \). What we call quantile counterpart of the process (8) is then given by

\[
\sqrt{T} \left( \int_0^1 \hat{L}_T^q(t) \, dt - \frac{1}{3} s^2 (1 - s)^2 (C_1(q) - C_2(q))^2 \right), \tag{9}
\]

where \( \hat{L}_T(t) \) from (8) is replaced by its quantile version \( \hat{L}_T^q(t) := (\hat{U}_T(s, q))^2 \) for \( q \in [0, 1]^N \) fixed. Accordingly, the test statistic \( \hat{\kappa}_T^q \) is then defined as

\[
\hat{\kappa}_T^q := \int_0^1 \hat{L}_T^q(t) \, dt. \tag{10}
\]

Since the limit distributions of the processes (8) and (9) are not known in case of unknown marginals we suggest a bootstrap procedure. The null hypothesis will be rejected if the expression in (8) or (9) is greater than the value of the corresponding quantile which can be obtained by applying the bootstrap procedure presented in Section 3. The test holds the size level if the fixed adjustable threshold \( \Delta \) is chosen as \( \|C_1(u) - C_2(u)\|_{L^2} \) or for the quantile case \( |C_1(u) - C_2(u)| \). For \( \Delta \) smaller than this threshold the test is oversized while a larger \( \Delta \) results in a lower rejection rate. In the application later on we haven chosen \( q = (q, \ldots, q) \in [0, 1]^N \) with \( q \in [0, 1] \) so that it maximizes the copula difference and set \( \Delta = |C_1(q) - C_2(q)| \). An \( \Delta \) chosen in this way can be used, for example, to assess the extent of a crisis.

Our Monte Carlo simulations below confirm that the bootstrap results in a reasonably sized and powered testing procedure. Note that for the bootstrap we consider the \( L^2 \)-norm, but this can be easily adjusted to the quantile version simply by interchanging the \( L^2 \)-norm with the absolute value \( |\cdot| \) for fixed \( q \in [0, 1]^N \).

### 3. Bootstrap and Testing Procedure

The bootstrap is based on the natural estimators of the respective terms of the process (8) and (9). We assume that our sample \( \{X_i\}_{i=1}^T \) is serially independently distributed or residual data from pre-estimated time series models e.g. GARCH adjusted data. Further, \( \{X_i\}_{i=1}^T \) is compounded of \( \{X_i\}_{i=sT}^{(sT)} \) and \( \{X_i\}_{i=[sT]+1}^T \), such that there is only one breakpoint location in \( [sT], s \in (0, 1) \) and \( \{X_i\}_{i=sT}^{(sT)} \sim C_1(F(X)) \) and \( \{X_i\}_{i=[sT]+1}^T \sim C_2(F(X)) \). Then, the bootstrap procedure suggests the following course of action:

i) Estimate the breakpoint location \( [sT] \) by \( \hat{s}T \), where \( \hat{s} \) is determined by

\[
\hat{s} := \arg\max_{s \in (0, 1)} \|\hat{U}_T(s, u)\|_{L^2}. \tag{11}
\]

Sample separately with replacement from \( \{X_i\}_{i=1}^{[sT]} \) and \( \{X_i\}_{i=[sT]+1}^T \) to obtain \( B \) bootstrap samples \( \{X_i^{(b)}\}_{i=1}^T \), for \( b = 1, \ldots, B \).
ii) Estimate the break point location \(|\hat{s}_b T\)| for each bootstrap sample \(\{X_i^{(b)}\}_{i=1}^T\), for \(b = 1, \ldots, B\), using adjusted (11).

iii) Estimate the copula difference \(\Delta^b_C = \|\hat{C}^{t_1:t_2}_b(u) - \hat{C}^{t_1:t_2}_b(1:T)(u)\|_{L^2}\) for each bootstrap sample \(\{X_i^{(b)}\}_{i=1}^T\), for \(b = 1, \ldots, B\), where \(\hat{C}^{t_1:t_2}_b\) is the empirical copula estimate with sequentially estimated marginals, using the data from \(t_1\) to \(t_2\).

iii) Calculate the bootstrap versions of the centred expressions (8) or (9)

\[
K^{(b)} := \sqrt{T} \left( \int_0^1 \hat{L}^{s^b}_T(t) dt - \frac{1}{3} \hat{s}_b^2 (1 - \hat{s}_b)^2 \Delta^b_C \right),
\]

with \(\hat{L}^{s^b}_T(t) := \|\hat{U}^{s^b}_T(s, u)\|_{L^2}\), where \(\hat{U}^{s^b}_T(s, u)\) is the bootstrap analogue of (3), using \(\{X_i^{(b)}\}_{i=1}^T\).

iv) Compute \(B\) versions of \(K^{(b)}\) and determine the critical value \(c\) such that

\[
\frac{1}{B} \sum_{b=1}^B I\{K^{(b)} > c\} = q,
\]

where \(q \in (0, 1)\).

With the above described bootstrap procedure we can calculate critical values for (8) and (9). The testing procedure is as follows: We reject the null of no relevant change \(\|C_1(u) - C_2(u)\|_{L^2} \leq \Delta\) if

\[
\hat{\kappa}_T > \frac{\hat{s}^2 (1 - \hat{s})^2}{3} \Delta^2 + \frac{b_{1-\alpha}}{\sqrt{T}},
\]

where \(b_{1-\alpha}\) is the \(1 - \alpha\) quantile of the bootstrap distribution.

The bootstrap and testing procedure can be easily adapted for the quantile case by adapting step i) - iii). Note, the test given in equation (12) is an exact level \(\alpha\) test if \(\Delta\) is chosen as the copula difference \(\|C_1(u) - C_2(u)\|_{L^2}\) or \(\|C_1(q) - C_2(q)\|\). Otherwise the size is smaller than \(\alpha\). Thus, \(\hat{\kappa}_T\) converges weakly to a degenerated random variable if the copula difference is equal to zero (no break point). Consequently, the level of the proposed tests have practically size zero, whereas classical stationarity tests hold the asymptotic \(\alpha\)-level. Thus, the power of the classical tests is usually larger than the power of the relevant change tests considered here. For practitioners we suggest to run a classical test first, e.g. Bücher and Ruppert (2013) for the case of known marginals and Bücher et al. (2014) in the case of sequentially estimated marginals. If the test rejects the null of stationarity, i.e. the copula difference is significantly larger than zero, estimate the break fraction and apply the proposed relevant change test.

4. QUANTILE- AND \(L^2\)-SIMULATIONS

In this section we want to analyze finite sample properties of our proposed relevant testing procedure, where we simulate multivariate data up to dimension \(N = 30\) using a factor copula...
model following Oh and Patton (2017). We consider both serially independently distributed and residual data.

4.1. Serially Independently Distributed Data

In this subsection we conduct two major Monte Carlo simulations. First, we consider the following simple DGP

\[ X_t = \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = N_2(0, \Sigma_t(\rho)) , \]

where \( N_2(0, \Sigma_t(\rho)) \) with \( t = 1, ..., T \) describes the bivariate normal distribution with expectation vector zero and covariance matrix \( \Sigma_t(\rho) = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \) and \( \rho \in [-1, 1] \). We set \( \rho \) equal to \(-0.3\) for \( t = 1, ..., \frac{T}{2} \) and \( \rho = 0.8 \) for \( t = \frac{T}{2} + 1, ..., T \). Thus, the breakpoint \( sT \) is chosen at \( \frac{T}{2} \).

The following size study presents both \( L_2 \)-norm based results and an analysis where we consider the specific point \( q = (0.6, 0.6) \). Note, the closer the quantile is to its boundaries, i.e. 0 or 1, the more observations are needed. Critical values of our tests are computed using the bootstrap algorithms from Sections 3 with \( B = 300 \) bootstrap replications. The tests are performed at the \( \alpha = 0.05, 0.1 \) significance level using 301 Monte Carlo replications. The computations were implemented in Matlab, parallelized and performed using CHEOPS, a scientific High Performance Computer at the Regional Computing Center of the University of Cologne (RRZK).

Table 1 presents the results of the relevant change tests under the null with \( \Delta \) chosen as the estimated copula difference \( |C_1(q) - C_2(q)| \), where \( C_1 \) and \( C_2 \) are estimated by the consistent copula estimator

\[ \hat{C}(u) = \frac{1}{t_2 - t_1} \sum_{i=t_1}^{t_2} \mathbb{I}\{\hat{F}_1^{t_1:t_2}(X_{i1}) \leq u_1, \ldots, \hat{F}_N^{t_1:t_2}(X_{iN}) \leq u_N\} , \]

using realizations \( \{X_1, \ldots, X_{\lfloor \hat{s}T \rfloor}\} \) and \( \{X_{\lfloor \hat{s}T \rfloor + 1}, \ldots, X_T\} \). The breakpoint \( \lfloor \hat{s}T \rfloor \) is estimated by

\[ \hat{s} := \arg\max_{s \in (0,1)} |\hat{U}_T(s, q)| . \]

Table 2 reports the results of the relevant change tests under the null, where the functional difference between the copulas is determined by the \( L_2 \)-norm. Similar to the quantile case we consider for the size analysis \( \Delta := \|C_1(u) - C_2(u)\|_{L^2} \) and accordingly \( \hat{s} := \arg\max_{s \in (0,1)} \|\hat{U}_T(s, u)\|_{L^2} \).

Collectively, the tests show good size properties and converges to the predetermined rejection level \( \alpha \) if \( T \) gets larger.

For the power analysis we use the quantile based test and consider two different scenarios.
Table 1: Size using quantile version

<table>
<thead>
<tr>
<th></th>
<th>$T = 300$</th>
<th>$T = 500$</th>
<th>$T = 750$</th>
<th>$T = 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_{95}$</td>
<td>0.060</td>
<td>0.060</td>
<td>0.043</td>
<td>0.053</td>
</tr>
<tr>
<td>$q_{90}$</td>
<td>0.122</td>
<td>0.113</td>
<td>0.099</td>
<td>0.103</td>
</tr>
</tbody>
</table>

Table 1 reports the rejection rate of the relevant change test for data generated with the DGP described in (13) using $B = 300$ bootstrap replications. The copula difference is evaluated at $q = (0.6, 0.6)$. In total, we conducted 301 Monte Carlo replications.

Table 2: Size using the $L^2$-norm

<table>
<thead>
<tr>
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Table 2 reports the rejection rate of the relevant change test for data generated with the DGP described in (13) using $B = 300$ bootstrap replications. The copula difference is determined using the $L^2$-norm. In total, we conducted 301 Monte Carlo replications.

In the first scenario we keep $\Delta$ fix and vary $\rho$ in the DGP (13). In the second scenario we vary $\Delta$ and keep the DGP (13) fixed.

The upper panel of table 3 depicts the first scenario. In this case we determine $\Delta_0$ as the copula difference at the point $q = (0.6, 0.6)$ generated by the DGP (13) with $\rho = -0.3$ before and $\rho = 0.8$ after the break point at $T/2$. We now vary $\rho \in \{-0.4, -0.5, -0.6, -0.7\}$ before the break point and the results of the rejection rate can be seen in the upper panel of table 3 for different sample sizes.

The lower panel of table 3 depicts the second scenario. After determining the quantile value under the null we decrease the tolerance $\Delta$ in the test (12) by $\Delta = d \cdot \Delta_0$ with $d \in \{0.95, 0.9, 0.85, 0.8\}$.

Note, that in both cases the rejection rate of the relevant change test holds the size level $\alpha$ and the rejection rate tends to 1 for increasing sample size $T$ and decreasing $d$ or $\rho$.

In the second major MC simulation, we consider our data to be jointly distributed with a one factor copula model following Oh and Patton (2017), where the marginal distributions are in general unknown and the copula is implied by the following factor structure

$$X_t = [X_{1t}, \ldots, X_{Nt}]' = \beta_t Z + q,$$

with $\beta_t = \beta_t \cdot (1, \ldots, 1)'$ is a parameter vector of size $N$, $Z \overset{i.i.d.}{\sim} \text{Skew t} (\nu^{-1}, \lambda)^4$ and $q = [q_{1t}, \ldots, q_{Nt}]'$ with $q_{it} \overset{i.i.d.}{\sim} t (\nu^{-1})$ for $i = 1, \ldots, N$ and $t = 1, \ldots, T$. We fix $\nu^{-1} = 0.25$ and $\lambda = -0.5$, such that our model is parametrized by the single factor loading $\theta_t = \beta_t$ for

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$^4$As in Oh and Patton (2017) this refers to the skewed t-distribution by Hansen (1994).
Table 3 reports the rejection rate of the quantile relevant change test for data generated with the DGP described in (13) using $B = 300$ bootstrap replications. The copula difference is evaluated at $q = (0.6, 0.6)$. Varying $\Delta = d \cdot \Delta_0$, where $d = \{0.95, 0.9, 0.85, 0.8\}$ (upper panel) and $\rho = \{-0.4, -0.5, -0.6, -0.7\}$ in $\Sigma_t(\rho)$ (lower panel) for $t = 1, \ldots, T$. In total, we conducted 301 Monte Carlo replications.

To analyze the power of the test, we consider two different scenarios using the quantile based test. First, we set a fixed $\Delta$ while we increase the copula difference by increasing the parameter $\theta_1$ after the break. Second, we keep the parameter values $\theta_0 = 1$ and $\theta_1 = 2$ fixed and decrease $\Delta$, while the starting point for $\Delta$ is equal to the implied copula difference at $q = 0.6 \cdot (1, \ldots, 1)'$. For the power analysis we consider two different scenarios using the quantile based test. First, we set a fixed $\Delta$ while we increase the copula difference by increasing the parameter $\theta_1$ after the break. Second, we keep the parameter values $\theta_0 = 1$ and $\theta_1 = 2$ fixed and decrease $\Delta$, while the starting point for $\Delta$ is equal to the implied copula difference at $q = 0.6 \cdot (1, \ldots, 1)'$.

Table 6 reports the rejection rate of the test (12) using the 95%—quantile of the proposed...
Table 4: Size using the $L^2$-norm

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<tr>
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<th>$T = 300$</th>
<th>$T = 500$</th>
<th>$T = 750$</th>
<th>$T = 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 2$</td>
<td>$q_{95}$</td>
<td>0.0300</td>
<td>0.0350</td>
<td>0.0399</td>
</tr>
<tr>
<td></td>
<td>$q_{90}$</td>
<td>0.0749</td>
<td>0.0817</td>
<td>0.1115</td>
</tr>
<tr>
<td>$N = 3$</td>
<td>$q_{95}$</td>
<td>0.0266</td>
<td>0.0483</td>
<td>0.0417</td>
</tr>
<tr>
<td></td>
<td>$q_{90}$</td>
<td>0.0599</td>
<td>0.0998</td>
<td>0.1183</td>
</tr>
<tr>
<td>$N = 5$</td>
<td>$q_{95}$</td>
<td>0.0200</td>
<td>0.0399</td>
<td>0.0732</td>
</tr>
<tr>
<td></td>
<td>$q_{90}$</td>
<td>0.0433</td>
<td>0.0849</td>
<td>0.1281</td>
</tr>
</tbody>
</table>

Table 4 shows the rejection rate of the relevant change test for the DGP (16) using $B = 300$ bootstrap replication. In total, we conducted 301 Monte Carlo repetitions.

Table 5: Size using quantile version

<table>
<thead>
<tr>
<th></th>
<th>$T = 300$</th>
<th>$T = 500$</th>
<th>$T = 750$</th>
<th>$T = 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 2$</td>
<td>$q_{95}$</td>
<td>0.0601</td>
<td>0.0432</td>
<td>0.0460</td>
</tr>
<tr>
<td></td>
<td>$q_{90}$</td>
<td>0.1419</td>
<td>0.1030</td>
<td>0.1299</td>
</tr>
<tr>
<td>$N = 3$</td>
<td>$q_{95}$</td>
<td>0.0799</td>
<td>0.0729</td>
<td>0.0599</td>
</tr>
<tr>
<td></td>
<td>$q_{90}$</td>
<td>0.1548</td>
<td>0.1347</td>
<td>0.1169</td>
</tr>
<tr>
<td>$N = 5$</td>
<td>$q_{95}$</td>
<td>0.0370</td>
<td>0.0380</td>
<td>0.0519</td>
</tr>
<tr>
<td></td>
<td>$q_{90}$</td>
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<td>0.0959</td>
<td>0.1319</td>
</tr>
<tr>
<td>$N = 30$</td>
<td>$q_{95}$</td>
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<td>0.0380</td>
<td>0.0490</td>
</tr>
<tr>
<td></td>
<td>$q_{90}$</td>
<td>0.0829</td>
<td>0.0999</td>
<td>0.0859</td>
</tr>
</tbody>
</table>

Table 5 reports the rejection rate of the relevant change test for data generated with the DGP described in (16) using $B = 300$ bootstrap replications. The copula difference is evaluated at $q = 0.6 \cdot (1,\ldots,1)'$. In total, we conducted 301 Monte Carlo repetitions.

bootstrap distribution in Section 3. The first column depicts the rejection rate under the null hypothesis. The values of the other columns are obtained by increasing the corresponding copula parameter $\theta_1 \in \{2.2, 2.4, 2.6, 2.8\}$, while $\Delta$ remains fixed to the initial copula difference, i.e. $\theta_0 = 1$ and $\theta_1 = 2$.

Table 6 illustrates that the power of the test (12) increases not only if $T$ but also if the cross sectional dimension $N$ increases. For example, the scenario $N = 30$, $T = 750$ and $\theta_1 = 2.6$ always rejects the null hypothesis, i.e. the rejection rate is equal to 1. This is also expected, since we increase the parameter in the factor copula model (16) for each component. Consequently, the error is effectively added up which leads to the gain in power.

Finally, table 7 analyses the rejection rate if $\Delta$ decreases while the copula difference remains fixed. The value $\Delta_0$ is equal to the copula difference computed at the point $q = 0.6 \cdot (1,\ldots,1)'$. 

11
Table 6: Power Analysis varying $\theta$ of the DGP (16)

<table>
<thead>
<tr>
<th></th>
<th>$\theta_1 = 2.0$</th>
<th>$\theta_1 = 2.2$</th>
<th>$\theta_1 = 2.4$</th>
<th>$\theta_1 = 2.6$</th>
<th>$\theta_1 = 2.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 2$</td>
<td>$T = 300$</td>
<td>0.0603</td>
<td>0.4585</td>
<td>0.7043</td>
<td>0.7973</td>
</tr>
<tr>
<td></td>
<td>$T = 500$</td>
<td>0.0432</td>
<td>0.4817</td>
<td>0.7475</td>
<td>0.8704</td>
</tr>
<tr>
<td></td>
<td>$T = 750$</td>
<td>0.0460</td>
<td>0.5548</td>
<td>0.8073</td>
<td>0.9336</td>
</tr>
<tr>
<td></td>
<td>$T = 1000$</td>
<td>0.0619</td>
<td>0.5714</td>
<td>0.8704</td>
<td>0.9668</td>
</tr>
<tr>
<td>$N = 3$</td>
<td>$T = 300$</td>
<td>0.0799</td>
<td>0.4917</td>
<td>0.7010</td>
<td>0.8605</td>
</tr>
<tr>
<td></td>
<td>$T = 500$</td>
<td>0.0729</td>
<td>0.4419</td>
<td>0.7475</td>
<td>0.8937</td>
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<tr>
<td></td>
<td>$T = 750$</td>
<td>0.0599</td>
<td>0.5548</td>
<td>0.8073</td>
<td>0.9734</td>
</tr>
<tr>
<td></td>
<td>$T = 1000$</td>
<td>0.0569</td>
<td>0.6445</td>
<td>0.9369</td>
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<td>$T = 300$</td>
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<td>0.4219</td>
<td>0.7010</td>
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<tr>
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<td>$T = 500$</td>
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<td>0.4983</td>
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<tr>
<td></td>
<td>$T = 750$</td>
<td>0.0519</td>
<td>0.5914</td>
<td>0.9468</td>
<td>0.9967</td>
</tr>
<tr>
<td></td>
<td>$T = 1000$</td>
<td>0.0509</td>
<td>0.6678</td>
<td>0.9369</td>
<td>1.0000</td>
</tr>
<tr>
<td>$N = 30$</td>
<td>$T = 300$</td>
<td>0.0330</td>
<td>0.5615</td>
<td>0.8007</td>
<td>0.9269</td>
</tr>
<tr>
<td></td>
<td>$T = 500$</td>
<td>0.0380</td>
<td>0.5482</td>
<td>0.9203</td>
<td>0.9867</td>
</tr>
<tr>
<td></td>
<td>$T = 750$</td>
<td>0.0490</td>
<td>0.6777</td>
<td>0.9701</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>$T = 1000$</td>
<td>0.0609</td>
<td>0.7841</td>
<td>0.9900</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 6 reports the rejection rate of the quantile relevant change test for data generated with the DGP described in (16) using $B = 300$ bootstrap replications. The copula difference is evaluated at $q = 0.6 \cdot (1, \ldots, 1)'$ varying $\theta_1 \in \{2.0, 2.2, 2.4, 2.6, 2.8\}$ with $\theta_0 = 1$ in the DGP (16). In total, we conducted 301 Monte Carlo repetitions.

Now, we decrease $\Delta$ stepwise, i.e. $\Delta = d \cdot \Delta_0$ with $d \in \{0.95, 0.9, 0.85, 0.8, 0.75\}$. Table 7 shows, that the rejection rate tends to 1 if $T$ increases.
Table 7: Power Analysis varying $\Delta$ of the DGP (16)

<table>
<thead>
<tr>
<th>$N$</th>
<th>$T$</th>
<th>$\Delta = \Delta_0$</th>
<th>$\Delta = 0.95 \cdot \Delta_0$</th>
<th>$\Delta = 0.9 \cdot \Delta_0$</th>
<th>$\Delta = 0.85 \cdot \Delta_0$</th>
<th>$\Delta = 0.8 \cdot \Delta_0$</th>
<th>$\Delta = 0.75 \cdot \Delta_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>300</td>
<td>0.0603</td>
<td>0.1794</td>
<td>0.3621</td>
<td>0.5681</td>
<td>0.7243</td>
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</tr>
<tr>
<td></td>
<td>500</td>
<td>0.0432</td>
<td>0.1894</td>
<td>0.3654</td>
<td>0.5482</td>
<td>0.7176</td>
<td>0.9435</td>
</tr>
<tr>
<td></td>
<td>750</td>
<td>0.0460</td>
<td>0.1462</td>
<td>0.3455</td>
<td>0.5581</td>
<td>0.7375</td>
<td>0.9468</td>
</tr>
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<td></td>
<td>1000</td>
<td>0.0619</td>
<td>0.2193</td>
<td>0.4286</td>
<td>0.6412</td>
<td>0.7973</td>
<td>0.9767</td>
</tr>
<tr>
<td>3</td>
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<td>0.1666</td>
<td>0.3688</td>
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<td>0.9468</td>
</tr>
<tr>
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<td>0.0729</td>
<td>0.1462</td>
<td>0.3156</td>
<td>0.5349</td>
<td>0.7276</td>
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<tr>
<td></td>
<td>750</td>
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<td>0.1993</td>
<td>0.4083</td>
<td>0.6379</td>
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<td>0.9668</td>
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<tr>
<td></td>
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<td>0.5944</td>
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<td>0.9834</td>
</tr>
<tr>
<td>5</td>
<td>300</td>
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<td>0.1462</td>
<td>0.2857</td>
<td>0.5050</td>
<td>0.6877</td>
<td>0.9007</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>0.0380</td>
<td>0.1628</td>
<td>0.3488</td>
<td>0.5581</td>
<td>0.7508</td>
<td>0.9405</td>
</tr>
<tr>
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<td>750</td>
<td>0.0519</td>
<td>0.1863</td>
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<td>0.1694</td>
<td>0.4319</td>
<td>0.7342</td>
<td>0.9169</td>
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<tr>
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</tr>
<tr>
<td></td>
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<tr>
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<td>0.4950</td>
<td>0.7508</td>
<td>0.9136</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 7 reports the rejection rate of the quantile relevant change test for data generated with the DGP described in (16) using $B = 300$ bootstrap replications and 301 Monte Carlo repetitions. The copula difference is evaluated at $q = 0.6 \cdot (1, \ldots, 1)'$, while $\Delta = d \cdot \Delta_0$ with $d \in \{0.95, 0.9, 0.85, 0.8, 0.75\}$. 


4.2. Residual Data

In this subsection we consider residual data $X_t$ from pre-estimated time series models for $t = 1, \ldots, T$. For our simulation we consider a GARCH(1,1) model, i.e.

$$r_{it} = \sigma_{it} X_{it}$$
$$\sigma_{it}^2 = \alpha_0 + \alpha_1 r_{i,t-1}^2 + \beta_1 \sigma_{i,t-1}^2$$

for $i = 1, \ldots, N$ and $t = 1, \ldots, T$. To get serial correlated data we first simulate residual data using the factor copula model (16) with a break constructed at $T/3$ and $\theta_0 = 1$ and $\theta_1 = 2$. Then we transform the residual data in serial correlated data $r_{it}$ using the GARCH(1,1) model with fixed parameter values $\alpha_0 = \frac{1}{15}, \alpha_1 = \frac{1}{15}$ and $\beta_1 = \frac{1}{3}$ for $i = 1, \ldots, N$ and $t = 1, \ldots, T$. With the simulated serial correlated data $r_{it}$ we estimate the time series models using a GARCH(1,1) model and determine the residual data $X_{it}$ for $i = 1, \ldots, N$ which is used to perform the test. We vary the sample size $T = 1000, 2000, 4000$ and cross sectional dimension $N = 3, 5, 10$. The results can be seen in table 8, which indicates, that the test using residual data holds the size level.

Table 8: Size using quantiles GARCH-data

<table>
<thead>
<tr>
<th>Copula with sequential estimated marginals</th>
<th>$T = 1000$</th>
<th>$T = 2000$</th>
<th>$T = 4000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 3$</td>
<td>$q_{95}$</td>
<td>0.0485</td>
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</tr>
<tr>
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<td>$q_{90}$</td>
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<td>0.1284</td>
</tr>
<tr>
<td>$N = 5$</td>
<td>$q_{95}$</td>
<td>0.0485</td>
<td>0.0599</td>
</tr>
<tr>
<td></td>
<td>$q_{90}$</td>
<td>0.1141</td>
<td>0.1369</td>
</tr>
<tr>
<td>$N = 10$</td>
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<td>0.0485</td>
</tr>
<tr>
<td></td>
<td>$q_{90}$</td>
<td>0.1098</td>
<td>0.0984</td>
</tr>
</tbody>
</table>

Table 8 reports the rejection rate of the relevant change test, where residual data from pre-estimated GARCH(1,1) models is considered. The copula difference is evaluated at $q = 0.6 \cdot (1, \ldots, 1)'$. In total, we conducted $B = 300$ bootstrap replications and 701 Monte Carlo repetitions.

5. APPLICATION

In this section we apply the quantile based test to a multivariate data set of cross-sectional dimension $N = 30$. We use daily aggregated stock log-returns over a time span ranging from October 2003 to July 2015 from the German DAX30, implying $T = 3000$ and $N = 30$. First, we estimate for a possible break point location in our data set, using (15), with the quantile $q = 0.93 \cdot (1, \ldots, 1)'$ and we receive $[\hat{s} T] = 1102$ (18.01.2008), cf. the black solid line in figure 1. Note, that we set the quantile $q := (q, ..., q) \in [0, 1]^N$ with $q \in [0, 1]$ such that it
maximizes the copula difference over all $q \in [0, 1]$, i.e. $q = \arg\max_{q \in [0, 1]} |C_1(q) - C_2(q)|$. One possible reason for the estimated break point can be the last financial crisis whose beginning can be traced back to the summer of 2006. Figure 1 shows the value of the empirical copula (cf. Equation (14)), evaluated at $q = 0.93 \cdot (1, \ldots, 1)'$, computed in a rolling window of size 300. Moreover, figure 1 reveals that the empirical copula increases strongly in the time of the last financial crisis between the summer of 2007 and the end of 2008, which indicates a strong change of the overall copula function at $u = 0.93 \cdot (1, \ldots, 1)'$. We receive $\hat{\kappa}_T = 1.9423 \cdot 10^{-4}$ and a 0.95-quantile value $b_{1-\alpha} = 2.5 \cdot 10^{-3}$, determined with the bootstrap from Section 3 at point $q = 0.93 \cdot (1, \ldots, 1)'$. The empirical copula difference $\delta := |\hat{C}_1(u) - \hat{C}_2(u)|$ is determined at the point $q = 0.93 \cdot (1, \ldots, 1)'$ and equals 0.1431. The relevant change test rejects the null hypothesis if

$$\hat{\kappa}_T > \frac{s^2(1 - s)^2}{3} \Delta^2 + \frac{b_{1-\alpha}}{\sqrt{T}},$$

where $b_{1-\alpha}$ is the $1 - \alpha$ quantile of the bootstrap distribution. We are interested in determining the smallest $\Delta$, such that the null hypothesis of no relevant change $|\hat{C}_1(u) - \hat{C}_2(u)| \leq \Delta$ can not be rejected. If we consider the full sample, we receive $\Delta = 0.6370 \cdot \delta = 0.0912$ which equals 63.7% of the true empirical copula difference.

Next, we estimate for a possible break point location in the pre-break data, i.e. October 2003 to January 2008. We estimated a break point at the 21.09.2006, which is depicted by the black dotted line in figure 5.1. One possible reason for the break point can be the start of the last financial crisis. Figure 5.1 reveals that the empirical copula increases significantly in summer 2006. We receive $\hat{\kappa}^\text{pre}_T = 1.5003 \cdot 10^{-5}$ and a 0.95-quantile value $b_{1-\alpha}^\text{pre} = 2.3116 \cdot 10^{-4}$, where the index $^\text{pre}$ indicates that we consider only the pre-break data. The copula difference $\delta^\text{pre} = |\hat{C}_1(u) - \hat{C}_2(u)|$ equals 0.0689 and the smallest $\Delta^\text{pre}$ where the null hypothesis of no relevant change can not be rejected, cf. (17), is $\Delta^\text{pre} = 0.338 \cdot \delta^\text{pre} = 0.0233$.

Analogously, we estimate for a possible break point in the post-break data starting in January 2008. We estimated a break point at the 06.01.2012, which is depicted by the black dashed line in figure 1. This corresponds to the last Euro crisis, which can be considered as a possible explanation. Figure 1 reveals that the empirical copula increases strongly during this period. We receive $\hat{\kappa}^\text{post}_T = 7.8865 \cdot 10^{-5}$ and a 0.95-quantile value $b_{1-\alpha}^\text{post} = 7.2361 \cdot 10^{-4}$. The copula difference $\delta^\text{post} = |\hat{C}_1(u) - \hat{C}_2(u)|$ equals 0.0864 at $q = 0.93 \cdot (1, \ldots, 1)'$ and the smallest $\Delta^\text{post}$ where the null hypothesis of no relevant change can not be rejected is $\Delta^\text{post} = 0.6360 \cdot \delta^\text{post} = 0.0549$.

To sum up, if $\Delta$ is chosen to be the smallest value such that the null hypothesis of no relevant change cannot be rejected, the testing procedure provides formula to determine $\Delta$ biuniquely. If $\Delta$ is chosen in the aforementioned way, it turns out that the $\Delta^\text{pre}$ of the pre-break data set is roughly the half of $\Delta^\text{post}$, which corresponds to the Euro crisis. The peak of the last financial crisis corresponds to 2008 which refers to $\Delta = 0.0912$, which is roughly the 1.7 of $\Delta^\text{post}$. Thus, the larger $\Delta$ the bigger are the effects of the corresponding crisis.
Figure 1: Value of the empirical copula defined in (14) evaluated at $q = 0.93 \cdot (1, \ldots, 1)$, computed in a rolling window of size 300. The estimated breakpoint, using (15), is displayed with the vertical black line (18.01.2008). Observed data between October 2003 and July 2015, implying $T = 3000$ and $N = 30$. 
6. CONCLUSION

In summary, the classical break point testing framework has two severe issues: On the one hand it considers a null which is theoretically never fulfilled and on the other hand any consistent test detects any arbitrary small change if the sample size is sufficiently large. Relevant change point analysis offers a way out.

We proposed a new non-parametric test for detecting relevant breaks in copula functions, where the hypothesis is of the form \( H_0 : \| C_1(u) - C_2(u) \| \leq \Delta \) versus \( H_1 : \| C_1(u) - C_2(u) \| > \Delta \) with \( \Delta \) a positive adjustable size to allow for difference in the copulas \( C_1 \) and \( C_2 \). Here, the norm in the hypothesis represents two different approaches: Either it measures the distance of the copulas given a certain value \( q \) or it equals the \( L^2 \)-norm.

As a starting point, we considered a natural CUSUM-type test statistic fitting to the underlined testing problem. For the estimation of the limiting distribution we constructed a new non-parametric bootstrap based on natural estimates of the constructed testing process, which is applicable in the case of unknown sequentially estimated marginal distributions.

In the case where the copula distance is measured at a given value \( q \) we considered simulated data up to cross sectional dimension \( N = 30 \). For the \( L^2 \)-norm we investigated the behavior of our test up to \( N = 5 \). The Monte Carlo simulation shows considerable size and power properties for both serially independent and residual data.

In our empirical application we analysed the German DAX30 data of cross sectional dimension \( N = 30 \) between October 2003 and July 2015. Here, \( \Delta \) is interpreted as the smallest admissible copula difference such that the relevant change hypothesis is not rejected. Every other choice of \( \Delta \) that is smaller leads to a rejection of the null hypothesis. Cutting the empirical data into three parts leads to a detection of the start of the financial crisis in 2006, the peak of the financial crisis in 2008 and the Euro crisis in 2012.

It turns out that \( \Delta \) can be regarded not only as the upper bound of an admissible copula distance, but also as a size that measures the extent of a crisis.
REFERENCES


