Sequential Detection of Parameter Changes in Dynamic Conditional Correlation Models

K. Pape, P. Galeano and D. Wied
TU Dortmund, Universidad Carlos III de Madrid, Universität zu Köln

Abstract

A multivariate monitoring procedure is presented to detect changes in the parameter vector of the Dynamic Conditional Correlation (DCC) model. The procedure can be used to detect changes in both the conditional and unconditional variances as well as in the correlation structure of the model. The detector is based on the contributions of individual observations to the gradient of the quasi-log-likelihood function. More precisely, standardized derivations of quasi-log-likelihood contributions of points in the monitoring period are evaluated at parameter estimates calculated from a historical period. The null hypothesis of a constant parameter vector is rejected if these standardized terms differ too much from zero. Limit results are derived under both the null hypothesis and alternatives. Size and power properties of the procedure are examined in a simulation study. Finally, the behavior of the proposed monitoring scheme is illustrated with a group of asset returns.

Keywords: Dynamic Conditional Correlation model; Multivariate sequences; Online detection; Parameter changes; Threshold function.

JEL Classification: C12; C32; C58.

Acknowledgements: Financial support by Deutsche Forschungsgemeinschaft (SFB 823, project A1) is gratefully acknowledged.
1 Introduction

In recent years a lot of research has been focused on modeling volatilities and correlations as well as on testing for structural breaks. Research in the intersection between these fields is motivated by the importance of being informed about changes in the variances and covariances or in the parameters that determine these characteristics, as soon as possible after their occurrence. In particular, in financial applications, analysts may need the aforementioned information to construct optimal portfolios or to anticipate crises since volatilities and correlations tend to increase in turbulent market phases, see for instance Sandoval Jr. and De Paula Franca (2012) or Charles and Darné (2014).

While former monitoring methods for multiple asset returns often focus either on variances or correlations, see for instance, Wied and Galeano (2013) and Pape et al. (2016), among others, we aim at monitoring structural changes in both volatilities and correlations jointly. For this purpose, we consider the popular Dynamic Conditional Correlation (DCC) model by Engle (2002) and provide a method to monitor its parameters which steer the conditional and unconditional volatilities and correlations. So, in contrast to Wied and Galeano (2013) and Pape et al. (2016), who propose methods which do not use a specific model assumption, our approach is model-based. This could in principle lead to efficiency gains as long as the model assumption fits to the data.

If the parameters are not constant in the observed period, then parameter estimates based on the constancy assumption are no longer reliable, as they yield biased volatility and correlation forecasts. Furthermore, the sum of the estimated autoregressive parameters of the conditional variance converges to one in univariate Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models if a parameter change is ignored, see Hillebrand (2005). This result should be kept in mind while dealing with multiple return series.

In contrast to the technique of Aue et al. (2009), who use a retrospective method to detect changes in the covariance structure of multivariate time series, we prefer a sequential monitoring procedure. That is, based on an historical period of observations, we obtain new data points bit by bit and

---

1 This means that we do not consider models such as Audrino and Trojani (2011) in which time-varying parameters are included from the beginning on.
use them to refresh our detector in order to determine the presence of a changepoint in the model parameters as soon as possible once it has happened. Our approach is motivated by the work of Chu et al. (1996) and Berkes et al. (2004). First, Chu et al. (1996) suggest to use the information of a historical sample, that is entirely available when the monitoring starts and is assumed to be free from structural breaks. Then, a size controlled sequential test is developed to check the structural stability of linear models based on cumulated recursive residuals. Building on this procedure, Berkes et al. (2004) propose a sequential monitoring scheme to detect changes in the parameters of the GARCH model. For this purpose, a detector that depends on quasi-likelihood scores is used. The historical sample is used to estimate the model parameters, that are used to evaluate the contributions of the data from the monitoring period to the Gaussian quasi-log-likelihood (QLL) function. Hence, under the alternative of a parameter change in the monitoring period, it is expected that the absolute gradient contributions of post-break observations tend to infinity.

The procedure proposed in this paper is used to monitor changes in the parameters of the DCC model and can be seen as a multivariate extension of the monitoring scheme proposed by Berkes et al. (2004). Nonetheless, the extension is much more complex than it may seem. Models that allow for dynamic modelling of both the variances and correlations possess a far more complex structure than other multivariate extensions of the univariate GARCH model. The challenge of handling the model and its quasi-likelihood scores gets even more demanding if a multiplicative structure of the conditional covariance matrix is postulated as in the DCC model. Also, the DCC models and their properties are far less well investigated than univariate GARCH models and especially the classical GARCH model considered by Berkes et al. (2004). For the models with dynamic conditional correlations, important results like conditions for the existence and uniqueness of a stationary solution or for the existence of unconditional moments of higher order have just been proposed recently, see Fermanian and Malongo (2017), or remain to be established, which makes this type of model quite challenging in applications.

Even if we focus on the DCC model due to its enormous popularity for modeling multiple financial returns, the results of this paper may be extended to models with structure similar to the one of the DCC model of Engle (2002), e.g., the Constant Conditional Correlation (CCC) Model of
Bollerslev (1990), the Varying Conditional Correlation (VCC) model of Tse and Tsui (2002) and the Asymmetric Generalized Dynamic Conditional Correlation (AG-DCC) model of Capiello et al. (2006), among others. On the contrary, the extension to other popular multivariate volatility models, e.g., the multivariate extensions of the GARCH models as proposed by Bollerslev et al. (1988) or the BEKK model proposed by Engle and Kroner (1995), that ensures the nonnegative definiteness of the conditional covariance matrix under milder conditions on the parameters, is more complex as the structure of these models is quite different than the structure of dynamic correlation models.

The rest of the paper is organized as follows. Section 2 briefly introduces the DCC model proposed by Engle (2002) and estimation of their parameters with quasi maximum likelihood. Section 3 presents necessary assumptions for the existence of a unique stationary solution of the DCC model and for the existence of higher moments. Also, asymptotic properties of the quasi maximum likelihood estimates (QMLE) of the model parameters are provided. Then, Section 4 presents the monitoring problem, proposes the monitoring scheme and establish their asymptotic properties. The performance of the procedure in finite time samples is investigated with the help of simulations and an application to real data in Sections 5 and 6, respectively. Finally, some concluding statements can be found in Section 7.

Additionally, the Supplementary Material of the paper contains several sections organized as follows. The first one provides a detailed presentation of the first and second order partial derivations of the contributions of individual observations to the QLL function of the model. The second section contains the proofs of the theorems and the proposition in Section 4. Finally, the third one summarizes the proofs of some additional calculation rules used along the paper.
2 The Dynamic Conditional Correlation Model

2.1 The Model and Basic Assumptions

Let \( \{ y_t, t \in \mathbb{Z}\} \) be a sequence of \( p \) dimensional random vectors, \( y_t = (y_{1t}, \ldots, y_{pt})' \), following a multivariate GARCH model given by

\[
y_t = H_t^{1/2} \epsilon_t \tag{2.1}
\]

where

\[
H_t = \text{Cov}(y_t|F_{t-1}) \tag{2.2}
\]

is the positive definite conditional covariance matrix of \( y_t \) given the information set \( F_{t-1} = \sigma \{ y_{t-1}, y_{t-2}, \ldots \} \) and \( \{ \epsilon_t, t \in \mathbb{Z} \} \) a standard white noise sequence in \( \mathbb{R}^p \), i.e. \( E(\epsilon_t) = 0_p, \text{Cov}(\epsilon_t) = I_p, \forall t \in \mathbb{Z} \), and the vectors \( \epsilon_t \) are mutually independent. In the following, \( 0_p, 0_{p \times p} \) and \( I_p \) denote the \( p \) dimensional vector of zeros, the \((p \times p)\) dimensional matrix of zeros, and the \((p \times p)\) dimensional identity matrix, respectively.

Of all the available specifications of the conditional covariance matrix \( H_t \), we focus on the one presented by Engle (2002). That is, we assume

\[
H_t = D_t R_t D_t \tag{2.3}
\]

with \( D_t = \text{diag}\left\{ h_{1t}^{1/2}, \ldots, h_{pt}^{1/2} \right\} \), where \( h_{it}, i = 1, \ldots, p \), are the individual variances, that can be specified for instance according to univariate GARCH(1,1) models:

\[
h_{it} = \omega_i + \alpha_i y_{it-1}^2 + \beta_i h_{it-1}, \quad i = 1, \ldots, p, \tag{2.4}
\]

for certain parameters \( \omega_i, \alpha_i \) and \( \beta_i \). Furthermore, \( R_t := \text{Cor}(y_t|F_{t-1}) \) is the conditional correlation matrix of \( y_t \), which can be decomposed as

\[
R_t = Q_t^* Q_t Q_t^*, \tag{2.5}
\]
where $Q_t$ is a $(p \times p)$ matrix that is recursively determined as

$$Q_t = (1 - \alpha - \beta) \bar{Q} + \alpha z_{t-1} z'_{t-1} + \beta Q_{t-1} \tag{2.6}$$

with $z_t = D_t^{-1} y_t$ the ‘standardized’ vectors. The parameters $\alpha$ and $\beta$ are nonnegative scalars, which satisfy $\alpha + \beta < 1$. $\bar{Q} = [\bar{q}_{ij}]_{i,j=1,...,p}$ is both the unconditional covariance and correlation matrix of $z_t$ in the special case of constant conditional correlations, see Aielli (2013). Motivated by this, we impose the restriction that the main diagonal elements are one, which is common in the literature. Consequently, the unknown parameters in the matrix $\bar{Q}$ are the entries of $\psi = \text{vec} \left( \bar{Q} \right) = (\bar{q}_{21}, \ldots, \bar{q}_{p,p-1})'$, where $\text{vec}(\cdot)$ is the operator that stacks the lower diagonal elements of a matrix into a vector. Finally, the normalizing matrix $Q_t^*$ is given by

$$Q_t^* := \text{diag} \left\{ \left[ Q_t \right]_{11}^{-1/2}, \ldots, \left[ Q_t \right]_{pp}^{-1/2} \right\}$$

where $[Q_t]_{ii}$ denotes the $i$-th main diagonal entry of the matrix $Q_t$.

In summary, the vector of parameters of the DCC model is given as

$$\theta = (\omega_1, \alpha_1, \beta_1, \ldots, \omega_p, \alpha_p, \beta_p, \alpha, \beta, \bar{q}_{21}, \ldots, \bar{q}_{p,p-1})'$$

which leads to a total number of $d := \frac{1}{2} (p + 1) (p + 4)$ unknown parameters in the model. Note that $\theta$ can be decomposed into $\theta = (\theta_1', \theta_2')'$, where

$$\theta_1 = (\omega_1, \alpha_1, \beta_1, \ldots, \omega_p, \alpha_p, \beta_p)' = (\phi_1', \ldots, \phi_p')'$$

with $\phi_j := (\omega_j, \alpha_j, \beta_j)'$, $j = 1, \ldots, p$, is the vector of variance parameters and

$$\theta_2 = (\alpha, \beta, \bar{q}_{21}, \ldots, \bar{q}_{p,p-1})' = (\alpha, \beta, \psi)'$$

is the vector of correlation parameters.

An important issue in multivariate models with dynamic variances and correlations is that the
positive definiteness of the conditional covariance matrix $H_t$ has to be guaranteed for all $t \in \mathbb{Z}$ almost surely. Proposition 2 in Engle and Sheppard (2001) gives sufficient conditions for this property. Particularly, the matrix $H_t$ as specified in (2.3)-(2.6), is positive definite for all $t \in \mathbb{Z}$ almost surely, if Assumption 2.1 is satisfied:

**Assumption 2.1.**

1. $\omega_i > 0$, $\forall i \in \{1, \ldots, p\}$.

2. $\alpha_i > 0$ and $\beta_i > 0$ with $\alpha_i + \beta_i < 1$, $\forall i \in \{1, \ldots, p\}$, see also Nelson and Cao (1992).

3. $h_{i0} > 0$, $\forall i \in \{1, \ldots, p\}$.

4. $\alpha > 0$ and $\beta > 0$ with $\alpha + \beta < 1$.

5. There exists $\delta_1 > 0$ with $\lambda_{\min}(\bar{Q}) > \delta_1$, where $\lambda_{\min}(\cdot)$ is the smallest eigenvalue of a square matrix.

6. There exists $\delta_2 > 0$ with $\lambda_{\max}(Q_t) \overset{a.s.}{<} \delta_2$, $\forall t \in \mathbb{Z}$, where $\lambda_{\max}(\cdot)$ is the largest eigenvalue of a square matrix.

Note that 2.1.(6) is not necessary to verify the positive definiteness of $R_t$, for all $t \in \mathbb{Z}$. This property is implied by the positive definiteness of $Q_t$ and Proposition 1 in Engle and Sheppard (2001). The fact that $Q_t$ is positive definite for all $t \in \mathbb{Z}$ is implied by the decomposition

$$Q_t = \frac{1 - \alpha - \beta}{1 - \beta} \bar{Q} + \alpha \sum_{n=0}^{\infty} \beta^n z_{t-n-1} z'_{t-n-1}$$

and 6.70.(a) in Seber (2008):

$$\lambda_{\min}(Q_t) \overset{a.s.}{\geq} \lambda_{\min} \left( \frac{1 - \alpha - \beta}{1 - \beta} \bar{Q} \right) + \lambda_{\min} \left( \alpha \sum_{n=0}^{\infty} \beta^n z_{t-n-1} z'_{t-n-1} \right) \overset{a.s.}{\geq} \frac{1 - \alpha - \beta}{1 - \beta} \lambda_{\min}(\bar{Q}) > \frac{1 - \alpha - \beta}{1 - \beta} \delta_1 > 0.$$ 

However, we use 2.1.(6) to get fixed boundaries for the positive eigenvalues of $R_t$, for all $t \in \mathbb{Z}$.

Note that

$$\max_{1 \leq i \leq p} [Q_t]_{ii} < \sum_{i=1}^{p} [Q_t]_{ii} = \sum_{i=1}^{p} \lambda_i (Q_t) \leq p \lambda_{\max}(Q_t) \overset{a.s.}{<} p \delta_2 \quad (2.7)$$
with $\lambda_{\text{max}}(Q_t) = \lambda_1(Q_t), \ldots, \lambda_p(Q_t) = \lambda_{\text{min}}(Q_t)^{a.s.} > 0$, the ordered eigenvalues of $Q_t$. Furthermore, we have

$$
\min_{1 \leq i \leq p} [Q_t]_{ii} > \frac{1 - \alpha - \beta}{1 - \beta}.
$$

(2.8)

Hence, (2.7), (2.8) and 6.95. in Seber (2008) imply the following boundaries for the eigenvalues of $R_t$:

$$
\lambda_{\text{min}}(R_t)^{a.s.} \geq \lambda_{\text{min}}(Q^*_t)^2 \lambda_{\text{min}}(Q_t)^{a.s.} \geq \left( \max_{1 \leq i \leq p} [Q_t]_{ii} \right)^{-1} \frac{1 - \alpha - \beta}{1 - \beta} \delta_1 \geq \frac{1 - \alpha - \beta}{1 - \beta} \frac{\delta_1}{p \delta_2} \quad (2.9)
$$

$$
\lambda_{\text{max}}(R_t)^{a.s.} \leq \lambda_{\text{max}}(Q^*_t)^2 \lambda_{\text{max}}(Q_t)^{a.s.} \leq \left( \min_{1 \leq i \leq p} [Q_t]_{ii} \right)^{-1} \delta_2 \leq \frac{1 - \beta}{1 - \alpha - \beta} \delta_2 \quad (2.10)
$$

The bounds in (2.9) and (2.10) will be used extensively throughout the paper and the proof section of the Supplementary Material.

### 2.2 Estimation of the Model Parameters

Given an observed multivariate time series $y_1, \ldots, y_T$, the QMLE of $\theta$ is a consistent parameter estimator that is obtained by maximizing the Gaussian QLL function

$$
L_T(\hat{\theta}) := L_T(\hat{\theta}|y_1, \ldots, y_T) = \sum_{t=1}^{T} l_t(\hat{\theta}|y_1, \ldots, y_T),
$$

(2.11)

with individual QLL contributions

$$
l_t(\hat{\theta}) := l_t(\hat{\theta}|y_1, \ldots, y_T) = -\frac{1}{2} \left( p \cdot \log 2\pi + \log \det (H_t) + y_t'H_t^{-1}y_t \right).
$$

(2.12)

Direct computation of the QMLE in one step is computationally expensive even for moderate dimensions of $y_t$. Alternatively, Engle and Sheppard (2001) proposed a two-step QMLE estimator to reduce the calculation time\(^2\). The two-step one is based on maximizing the part of the likelihood that only depends on the volatility parameters, $\theta_1$, and, after plug-in the estimates of these parameters and of the matrix $\bar{Q}$, maximize the rest of the likelihood that will only depends on

\(^2\)There is a also a recent approach for estimating the model equation by equation (Francq and Zakoïan, 2016), which we have not used.
the correlation parameters, $\theta$. The step of estimating $Q$ is called variance targeting, but this does not work in general, so that this procedure is in general not consistent (Aielli, 2013). Indeed, preliminary simulations showed that the one step QMLE yields distinctly better estimates than the two step one for the considered parameters values. Hence, we use the one step QMLE for our simulations and applications.\footnote{Another possibility would have been to consider the cDCC-model proposed by Aielli (2013) and to use the estimator proposed there.} In the following, we denote this estimator calculated from a sample of $T$ observations as $\hat{\theta}_T$. Furthermore, $L(\cdot)$ is the limit of the QLL function $L_T(\cdot)$ for $T \to \infty$.

3 General Results on the DCC Model

In this section, we present necessary assumptions for, first, the existence of a unique stationary solution of the DCC model, second, the existence of higher moments of such solution and, third, asymptotic properties of the QMLE of the model parameters. All these assumptions are needed for proving asymptotic properties of our monitoring procedure later on. When applicable, we use the notation in Berkes et al. (2004). The exposition of this section is similar to Fermanian and Malongo (2017), who make use of writing the DCC model as a nonlinear Markov chain. Another approach would be to derive stationarity and invertibility properties for the dynamic correlation matrix by representing the DCC model as a vector random coefficient moving average process as in McAleer (2017).

Denote the number of unknown parameters in the constant matrix $\bar{Q}$ as $p^{-} := \frac{1}{2}p(p - 1)$. Analogously to Berkes et al. (2003) and Berkes et al. (2004), we assume there exist constants $0 < \underline{u} < \bar{u}$ and $0 < \rho < 1$, such that:

\[
U := \left\{ u : \max \{ t_1, \ldots, t_p, b, a + b, |q_1|, \ldots, |q_{p^-}| \} \leq \rho, \ \lambda_{\min} \left( F_{\bar{Q}}(u) \right) > \delta_1, \text{ and } \underline{u} < \min \{ x_1, s_1, t_1, \ldots, x_p, s_p, t_p, a, b \} \leq \max \{ x_1, s_1, t_1, \ldots, x_p, s_p, t_p, a, b \} \leq \bar{u} \right\}
\]

where $u = (x_1, s_1, t_1, \ldots, x_p, s_p, t_p, a, b, q_1, \ldots, q_{p^-})'$ is a generic element of the constrained parameter space $U$. In the following, we use the elements of $U$ to distinguish between the obtained
realisations of terms like $h_{it}$ or $Q_t$, which depend on the true underlying parameter vector $\theta$, on the one hand, and variants of these functions, that might have been realised for other parameter values, on the other hand. The functions $F_{\bar{Q}}(u)$ and $F_{Q_t}(u)$ are defined as follows:

**Definition 3.1.** Define for $i \in \{1, \ldots, p\}$, $t \in \{2, 3, \ldots\}$ and $u \in U$:

1. $w_{it}(u) := \frac{x_i}{1-t_i} + s_i \sum_{k=1}^{\infty} t_i^{k-1} y_{i,t-k} = \frac{x_i}{1-t_i} + s_i \sum_{k=0}^{\infty} t_i^{k} y_{i,t-k}$.  

2. $F_{D_t}(u) := \text{diag} \left\{ w_{1t}(u)^{1/2}, \ldots, w_{pt}(u)^{1/2} \right\}$.  

3. $F_{R_t}(u) := F_{Q_t}(u) F_{Q_t}(u)^	op$ with $F_{Q_t}(u) := \text{diag} \left\{ [F_{Q_t}(u)]_{11}^{-1/2}, \ldots, [F_{Q_t}(u)]_{pp}^{-1/2} \right\}$.

4. $F_{Q_t}(u) := \frac{1-a-b}{1-b} F_{\bar{Q}}(u) + a \sum_{k=1}^{\infty} b^{k-1} z_{t-k}(u) z'_{t-k}(u)$.  

5. $F_{\bar{Q}}(u) := \begin{pmatrix} 1 & q_1 & q_2 & \ldots & q_{p-1} \\ q_1 & 1 & q_p & \ldots & \vdots \\ \vdots & q_p & \ddots & q_{p-1} & \vdots \\ \vdots & \vdots & \ddots & 1 & q_{p-1} \\ q_{p-1} & \ldots & q_{p-1} & q_p & 1 \end{pmatrix}$.

Note that $z_t(u) = F_{D_t}(u)^{-1} y_t$ which implies $z_{it}(u) = \frac{w_{it}(u)}{\sqrt{w_{it}(u)}}$, $\forall i = 1, \ldots, p$.

To enable consistent parameter estimation, we assume throughout the paper:

**Assumption 3.1.** $\theta \in U$.

The QLL function in (2.11) can be written as a function of an arbitrary element of the parameter space $U$ as $L_T(u) = \sum_{t=1}^{T} l_t(u)$ with

$$l_t(u) = -\frac{1}{2} \left( p \cdot \log 2\pi + \sum_{i=1}^{p} \log w_{it}(u) + \log \det (F_{R_t}(u)) + z'_t(u) F_{R_t}(u)^{-1} z_t(u) \right).$$

(3.1)

The functions $w_{it}(u)$ and $F_{Q_t}(u)$ depend on an infinite past of observations. While the assumption of an infinite past may be appropriate in the context of theoretical considerations, only finitely many past observations can be obtained in practice. Hence, we also consider the following terms:
Definition 3.2. Define for $i \in \{1, \ldots, p\}$, $t \geq 2$ and $u \in U$:

(i) $\hat{w}_{it}(u) := \frac{x_i}{1 - t_i} + s_i \sum_{k=1}^{t-1} t_i^{k-1} y_{i,t-k}^2 = \frac{x_i}{1 - t_i} + s_i \sum_{k=0}^{t-2} t_i^k y_{i,t-k-1}^2$.

(ii) $\hat{F}_{D_t}(u) := \text{diag} \left\{ \hat{w}_{1t}(u)^{1/2}, \ldots, \hat{w}_{pt}(u)^{1/2} \right\}$.

(iii) $\hat{F}_{R_t}(u) := \hat{F}_{Q_t}(u) \hat{F}_{Q_t}(u)$ with $\hat{F}_{Q_t}(u) := \text{diag} \left\{ \left[ \hat{F}_{Q_t}(u) \right]_{i1}^{-1/2}, \ldots, \left[ \hat{F}_{Q_t}(u) \right]_{pp}^{-1/2} \right\}$.

(iv) $\hat{F}_{Q_t}(u) := \frac{1 - a - b}{1 - b} F_{Q}(u) + a \sum_{k=1}^{t-1} b^{k-1} \hat{z}_{t-k}(u) \hat{z}_{t-k}(u)$ with $\hat{z}_t(u) = \hat{F}_{D_t}(u)^{-1} y_t$.

(v) $\hat{L}_T(u) = \sum_{t=1}^T \hat{l}_t(u)$

with $\hat{l}_t(u) = -\frac{1}{2} \left( p \cdot \log(2\pi) + \sum_{i=1}^p \log \hat{w}_{it}(u) + \log \det \left( \hat{F}_{R_t}(u) + \hat{z}_t(u) \hat{F}_{R_t}(u)^{-1} \hat{z}_t(u) \right) \right)$.

3.1 Existence of a Unique Stationary Solution

To verify the existence of a stationary and unique solution satisfying the DCC model along the lines of Fermanian and Malongo (2017), some additional assumptions have to be imposed. More precisely, as in Fermanian and Malongo (2017), the model equations (2.1)-(2.6) can be written in autoregressive form as

$$X_t = T_t X_{t-1} + \zeta_t$$

(3.2)

with

$$X_t := \left( h_{1t}, \ldots, h_{pt}, y_{1t}^2, \ldots, y_{pt}^2, \text{vecl} (Q_t)'', \text{vecl} (z_t z_t')'' \right)'$$

$$\zeta_t := \left( \omega_1, \ldots, \omega_p, \omega_1 z_{1t}^2, \ldots, \omega_p z_{pt}^2, (1 - \alpha - \beta) \text{vecl} (\tilde{Q})'', \text{vecl} (z_t z_t')'' \right)'$$
and \( T_t = \begin{bmatrix}
\beta_1 & 0 & \alpha_1 & 0 & 0_{p \times p^-} & 0_{p \times p^-} \\
\vdots & \ddots & \ddots & 0_{p \times p^-} & 0_{p \times p^-} \\
0 & \beta_p & 0 & \alpha_p & 0_{p \times p^-} & 0_{p \times p^-} \\
\beta_1 z_{1t}^2 & 0 & \alpha_1 z_{1t}^2 & 0 & 0_{p \times p^-} & 0_{p \times p^-} \\
\vdots & \ddots & \ddots & 0_{p \times p^-} & 0_{p \times p^-} \\
0 & \beta_p z_{pt}^2 & 0 & \alpha_p z_{pt}^2 & 0_{p \times p^-} & 0_{p \times p^-} \\
0_{p^- \times p} & 0_{p^- \times p} & 0_{p^- \times p} & \beta I_{p^-} & \alpha I_{p^-} \\
0_{p^- \times p} & 0_{p^- \times p} & 0_{p^- \times p} & 0_{p^- \times p} & 0_{p^- \times p} \\
\end{bmatrix}.\)

Denote as \( \mathcal{I} := \{ I_t, t \in \mathbb{Z} \} \) the filtration that is adapted to the process \( \{ X_t, t \in \mathbb{Z} \} \) in (3.2). Thus, \( I_t = \sigma(X_t, X_{t-1}, \ldots) \) is the information set at time \( t \). In the following, consider the process \( \{ \eta_t, t \in \mathbb{Z} \} \) with \( \eta_t := R_t^{-1/2} z_t \). Conditional upon the information up to time \( t-1 \), \( \eta_t \) behaves like an innovation vector and will be called conditional innovation. Analogously to Assumptions A0 and U0 in Fermanian and Malongo (2017), we assume

**Assumption 3.2.** The process \( \{ \eta_t, t \in \mathbb{Z} \} \) possesses the Markov property with respect to the filtration \( \mathcal{I} \). In particular, we have

\[
E(\eta_t | I_{t-1}) = E(\eta_t | X_{t-1}), \forall t \in \mathbb{Z}.
\]

Furthermore, \( \{ \eta_t, t \in \mathbb{Z} \} \) is ergodic and stationary.

For the model under consideration, the Assumptions U1-U3 in Fermanian and Malongo (2017) can be reduced to

**Assumption 3.3.**

\[
\max_{1 \leq i \leq p} \alpha_i + \max_{1 \leq i \leq p} \beta_i < 1 \quad \text{and} \quad |\beta| < 1.
\]

Along the lines of Fermanian and Malongo (2017) and by the use of Assumption 3.2, the autoregressive equation (3.2) can be represented as a nonlinear Markov chain. They have shown that the strict stationarity of \( \{ X_t, t \in \mathbb{Z} \} \) and thus of the DCC process \( \{ y_t, t \in \mathbb{Z} \} \) can be obtained with
Tweedie’s Theorem, see Tweedie (1988), which implies the existence of a time invariant measure for the transition probabilities of the linear Markov chain given by (3.2).

In addition, the uniqueness of the stationary solution can be obtained under the following assumption, which is equivalent to (14) in Fermanian and Malongo (2017):

**Assumption 3.4.**

\[
E \left[ \ln \left( \beta^2 + \alpha^2 \frac{4(2p + 1)\sqrt{p}}{\sqrt{C_\lambda C_q}} \| \eta \|_2^2 \right) \right] < 0
\]

where \( C_\lambda = \frac{\lambda_{\min}((1-\alpha-\beta)\bar{Q})}{1-\beta^2} \) and \( C_q = \frac{(1-\alpha-\beta) \min_{1 \leq i \leq p} q_{ii}}{1-\beta^2} \) are constants.

### 3.2 The Existence of Moments

To maintain asymptotic properties of the method proposed in the following section, the existence of 8th moments and cross moments has to be established. This property is proved by Fermanian and Malongo (2017) under the following assumptions that are equivalent to Assumptions E1 and E2 in their work:

**Assumption 3.5.** For some multiplicative matrix norm \( \| \cdot \| \), we have:

\[
E \left[ \| \eta \|^8 \right] < \infty \quad \text{and} \quad \lambda_{\max} \left( \sup_{x \in \mathbb{R}^d} E \left[ \| \bigotimes_k T_t \| \big| X_{t-1} = x \right] \right) < 1,
\]

where \( \bigotimes_k \) is the \( k \)-fold Kronecker product of identical matrices.

**Assumption 3.6.** The law of \( \eta_t | X_{t-1} = x \) is absolutely continuous with respect to the Lebesgue measure. The conditional density of \( \eta_t \) is denoted by \( f_{\eta_t}(\cdot|x) \) for every \( x \in \mathbb{R}^d \) and \( t \in \mathbb{Z} \). The density is continuous in \( x \) for every \( \eta \in \mathbb{R}^p \) and \( t \in \mathbb{Z} \). Furthermore, there exists an integrable function \( F \), such that

\[
\sup_{t \in \mathbb{Z}, x \in \mathbb{R}^d} f_{\eta_t}(\eta|x) \leq F(\eta), \quad \text{for every} \ \eta \in \mathbb{R}^p.
\]

Moreover,

\[
\sup_{t \in \mathbb{Z}} E \left[ \| \eta_t \|^8 | X_{t-1} = x \right] \leq g(\|x\|),
\]
for some function $g(\cdot)$ that satisfies
\[
\lim_{\nu \to \infty} \frac{g(\nu)}{\nu^{\varepsilon^*}} = 0 \text{ for every } \varepsilon^* > 0.
\]

### 3.3 Asymptotic Results for the Parameter Estimators

Recall that we estimate the model parameters by maximizing the function $L_T(\hat{\theta})$. For obtaining asymptotic results on the estimators, we need an assumption on the limit of this function.

**Assumption 3.7.** The limit function $L(\cdot)$ has a unique maximum in $\theta$.

The consistency of $\hat{\theta}_T$ follows from Assumption 3.7 and the uniform convergence from $L_T(\cdot)$ to $L(\cdot)$ in $U$. The latter result can be obtained with Theorem A.2.2 in White (1994) whose conditions are fulfilled since $U$ is a compact set and $l_t(u)$ is ergodic and continuous in $u \in U$ for all $y_t$ and measurable in $y_t$ for all $u \in U$ given the results in Fermanian and Malongo (2017). We choose the dominating function as $\sup_{u \in U} |l_t(u)|$ and use the results in Section 2.1 in the Supplementary Material, which yield
\[
E \left[ \sup_{u \in U} l_0(u) \right] < \infty.
\]

To obtain the asymptotic normality of the estimator, we have to assume the following regularity conditions:

**Assumption 3.8.**

1. Let $A(\theta) := \lim_{T \to \infty} E \left[ \frac{1}{T} \frac{\partial^2 L_T(\theta)}{\partial \theta \partial \theta'} \bigg|_{\theta = \theta} \right]$ be a finite and nonsingular matrix and 
   \{\theta^*_T, T \in \mathbb{N}\} a sequence with $\theta^*_T \overset{p}{\to} \theta$ for $T \to \infty$. For $T \to \infty$, we have:
   \[
   \frac{1}{T} \frac{\partial^2 L_T(\theta)}{\partial \theta \partial \theta'} \bigg|_{\theta = \theta^*_T} \overset{p}{\to} A(\theta).
   \]

2. Let $B(\theta) := \lim_{T \to \infty} E \left[ \frac{1}{T} \frac{\partial L_T(\theta)}{\partial \theta} \frac{\partial L_T(\theta)}{\partial \theta'} \bigg|_{\theta = \theta} \right]$ be a finite and nonsingular matrix. For $T \to \infty$, we have:
   \[
   \frac{1}{\sqrt{T}} \frac{\partial L_T(\theta)}{\partial \theta} \overset{d}{\to} \mathcal{N}(0_d, B(\theta)).
   \]

Since the results in Section 2.2 in the Supplementary Material imply that the Hessian of $l_0(u)$ exists
and is continuous in all of the parameters in an open and convex neighborhood of \( \theta \), Theorem 4.1.3 in Amemiya (1985) yields:

\[
\sqrt{T} \left( \hat{\theta}_T - \theta \right) \overset{d}{\to} \mathcal{N} \left( 0_d, A(\theta)^{-1} B(\theta) A(\theta)^{-1} \right).
\]

4 The Monitoring Procedure

Let \( \theta_t \in \mathbb{R}^d \) be the parameter vector of the DCC model at time \( t \). Assume a historical period of length \( m \) that is not affected by any structural change, i.e.

**Assumption 4.1.** \( \theta_1 = \ldots = \theta_m \) with \( m \) a positive integer.

We are interested in testing the null hypothesis of a constant parameter vector

\[
H_0 : \quad \theta_t = \theta, \quad t = 1, \ldots, m, m + 1, \ldots
\]

against the alternative of a change in the vector of parameters at an unknown point in the monitoring period

\[
H_1 : \quad \theta_t = \begin{cases} 
\theta, & t = 1, \ldots, m, m + 1, \ldots, m + k^* - 1 \\
\theta^*, & t = m + k^*, m + k^* + 1, \ldots
\end{cases}
\]

with \( \theta = (\phi_1', \ldots, \phi_p', \alpha, \beta, \psi')' \) the parameter vector before and \( \theta^* = (\phi_1^*, \ldots, \phi_p^*, \alpha^*, \beta^*, \psi^*)' \) the parameter vector after the change, where \( \phi_i^* = (\omega_i^*, \alpha_i^*, \beta_i^*)' \), \( i = 1, \ldots, p \), and \( \psi^* = \text{vec} \left( \bar{Q}^* \right) \) with \( \bar{Q}^* = [\bar{q}_{ij}^*]_{i,j=1,\ldots,p} \). Note that the change takes place at the \( k^* \)-th point of the monitoring period which is the \( (m + k^*) \)-th point in the entire time series.

The following monitoring scheme works similarly to the univariate method proposed by Berkes et al. (2004). Denote as \( l_t'(u) \) the gradient of the QLL contributions \( l_t(u) \) with infinite past in (3.1) and \( \hat{l}_t'(u) \) the gradient with finite past in Definition 3.2. The explicit form of the gradient is given in the Supplementary Material. Note that when the transpose of a vector and the gradient can be confused, we use \( T \) to denote the transpose of a vector.

Since the asymptotics are carried out under the assumption of a growing length of the historical
period, it may be suitable to define some characteristics as the length of the monitoring period in dependence of $m$. This also allows for a more appropriate comparison of the simulation results for different lengths of the historical period. We denote the length of the monitoring period as $m_B$. Thus, $B$ indicates how long the monitoring period is compared to the historical period. Furthermore, for any $u \in U$, define

$$D(u) := E_t I'_t(u) I'_t(u)^T$$

and assume

**Assumption 4.2.** $D := D(\theta)$ is a finite and nonsingular matrix.

For an observed sample, we use the estimate

$$\hat{D}_m = \frac{1}{m} \sum_{t=1}^{m} \hat{I}_t(\hat{\theta}_m) \hat{I}_t(\hat{\theta}_m)^T$$

whose consistency can be shown with the following proposition:

**Proposition 4.1.** Under Assumptions 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 4.1 and 4.2, we have

$$\hat{D}_m \to D \text{ a.s.}$$

The monitoring procedure is based on the detector

$$V_k = \sum_{t=m+1}^{m+k} \hat{D}_m^{-\frac{1}{2}} \hat{I}_t(\hat{\theta}_m)$$

with stopping rule

$$\tau_m = \min \left\{ k \leq mB : |V_k| > m^{\frac{1}{2}} \left(1 + \frac{k}{m}\right) b\left(\frac{k}{m}\right) \right\}, \quad (4.1)$$

where $b(\cdot)$ is a threshold function and $|\cdot|$ the norm that yields the maximum absolute entry of a vector and matrix. If $\tau_m < \infty$, a change in the parameters is indicated at some time point between $m + 1$ and $m + \tau_m$. If the detector did not cross the threshold function in the monitoring period
and no changepoint could be detected, the set on the right side of (4.1) would be the empty set and \( \tau_m = \infty \) as \( \min \emptyset = \infty \). As in Berkes et al. (2004), some moderate conditions are imposed on the form of the threshold function \( b(\cdot) \):

**Assumption 4.3.** \( b(\cdot) \) is continuous on \((0, \infty)\) and \( \inf_{0 < t < \infty} b(t) > 0 \).

To avoid confusion with the model parameters, let \( \hat{\alpha} \in (0, 1) \) be the significance level for testing the null hypothesis of no parameter change versus the alternative hypothesis of a change during the monitoring period. Therefore, the threshold function \( b(\cdot) \) or at least the variable parts of the function should be chosen such that

\[
\lim_{m \to \infty} P_{H_0} \{ \tau_m < \infty \} = \hat{\alpha} \quad \text{and} \quad \lim_{m \to \infty} P_{H_1} \{ \tau_m < \infty \} = 1.
\]

Berkes et al. (2004) choose the threshold function \( b(\cdot) \) as a constant that is obtained via simulation. Preliminary simulations suggested that the empirical size of the proposed multivariate procedure depends strongly on the length of the monitoring period, that is on the parameter \( B \), just as in the univariate case presented by Berkes et al. (2004). To reduce this effect, we include the length of the monitoring period into the stopping rule (4.1). Moreover, we prefer a curved threshold function to the linear one that results from choosing \( b(\cdot) \) as a constant function. In detail, we use the one proposed by Horváth et al. (2004) and also used by Wied and Galeano (2013) among others, i.e.

\[
b(x) = \max \left\{ \left( \frac{x}{1 + x} \right)^{\gamma}, \varepsilon \right\}
\]

where \( \gamma \in [0, \frac{1}{2}) \) is a tuning parameter and \( \varepsilon > 0 \) a constant, that can be chosen arbitrarily small in applications. A larger value of \( \gamma \) results in a steeper threshold function that tends to detect early changes in the parameters with a higher probability. In contrast, a smaller value of the tuning parameter leads to a lower slope of the threshold function, which results in a higher probability to detect changes that occur later in the monitoring period.

Additionally, we scale this threshold function by multiplying a constant \( c = c(\hat{\alpha}) \) that is obtained via Monte Carlo simulations, such that the probability that the detector crosses the threshold
function in the monitoring period equals the theoretical size $\tilde{\alpha}$.

**Theorem 4.1.** Under $H_0$ and Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2 and 4.3, we have

$$\lim_{m \to \infty} P_{H_0} \{ \tau_m < \infty \} = \lim_{m \to \infty} P_{H_0} \left( \sup_{1 < k \leq mBm^{1/2}} \frac{|V_k|}{1 + \frac{k}{m}} b \left( \frac{k}{m} \right) \geq c \right) = P_{H_0} \left( \sup_{t \in (0,B]} \frac{|G(t)|}{1 + t} b(t) \geq c \right)$$

where $\{G(t) = (G_1(t), \ldots, G_d(t))', \ t \in [0, B]\}$ is a $d$-variate stochastic process whose component processes are $d$ independent mean zero Gaussian processes $\{G_j(t), t \in [0, B]\}$ with covariance function $E(G_j(k)G_j(l)) = \min \{k, l\} + kl$, for $j = 1, \ldots, d$, where $d$ is the number of parameters in the DCC model.

Along the lines of Berkes et al. (2004) or Galeano and Wied (2014) and denoting $\{W_i(t), \ t \in [0, \infty)\}$ for $i = 1, \ldots, d$ as $d$ independent one dimensional Standard Brownian Motions, we have that $|G(t)|$ possesses the same distribution as $\max_{1 \leq i \leq p} \left| (1 + t)W_i \left( \frac{t}{1 + t} \right) \right|$ for all $t \in \mathbb{Z}$, which yields

$$\sup_{t \in (0,B]} \frac{|G(t)|}{1 + t} b(t) = \sup_{t \in (0,B]} \max_{1 \leq i \leq d} \left| W_i \left( \frac{t}{1 + t} \right) \right| = \sup_{t \in (0,B]} \max_{1 \leq i \leq d} \left\{ \left( \frac{t}{1 + t} \right)^\gamma, \varepsilon \right\}$$

if $t = \frac{s}{1 + s}$ is substituted. Furthermore, choosing $\tilde{s} = \frac{sB}{1 + B}$ yields

$$\sup_{t \in (0,B]} \frac{|G(t)|}{1 + t} b(t) = \sup_{s \in (0,1]} \max_{1 \leq i \leq d} \left| W_i \left( \frac{sB}{1 + B} \right)^\gamma \right| = \left( \frac{B}{1 + B} \right)^{\frac{1}{2} - \gamma} \sup_{s \in (0,1]} \max_{1 \leq i \leq d} \left\{ s^\gamma, \varepsilon \left( \frac{1 + B}{B} \right)^\gamma \right\}.$$ 

Thus, we can use Monte Carlo simulations to obtain critical values $c = c(\tilde{\alpha})$ in dependence of the significance level $\tilde{\alpha}$ based on the equality

$$P_{H_0} \left( \frac{B}{1 + B} \right)^{\frac{1}{2} - \gamma} \sup_{s \in (0,1]} \max_{1 \leq i \leq d} \left\{ s^\gamma, \varepsilon \left( \frac{1 + B}{B} \right)^\gamma \right\} \geq c(\tilde{\alpha})$$
\[
1 - \left[ P_{H_0} \left( \sup_{s \in (0,1]} \max \{ s^\gamma, \varepsilon \left( \frac{1+B}{B} \right)^\gamma \} < \left( \frac{1+B}{B} \right)^{1/2-\gamma} \varepsilon (\tilde{\alpha}) \right) \right]^d = \tilde{\alpha}
\]

or alternatively
\[
P_{H_0} \left( \sup_{s \in (0,1]} \max \{ s^\gamma, \varepsilon \left( \frac{1+B}{B} \right)^\gamma \} < \left( \frac{1+B}{B} \right)^{1/2-\gamma} \varepsilon (\tilde{\alpha}) \right) = (1 - \tilde{\alpha})^{1/2}.
\]

Simulations showed that the critical values obtained by the use of the limit distribution of the detector yield infeasible high size distortions even for large-sized historical periods in finite samples. As a consequence the detector values tend to exceed the values of the scaled threshold function soon after the beginning of the monitoring period, whether a parameter change occurs or not. To extenuate the resulting size distortions, the critical values can be obtained via a parametric Bootstrap type procedure. Recall that \( \hat{\theta}_m \) is the estimate of the parameter vector calculated from the historical sample. We assume that the underlying DCC process features a similar behavior as the process determined by the parameters estimated from the historical period, if the latter one is sufficiently large. Hence, \( b_{BT} = 199 \) realizations of a DCC process whose structure is controlled by \( \hat{\theta}_m \) (and the innovations follow a multivariate standard normal distribution) are simulated and denoted as \( Y^{*\langle i \rangle} := \{ y_1^{\langle i \rangle}, \ldots, y_{m(B+1)}^{\langle i \rangle} \} \), for \( i \in \{1, \ldots, b_{BT}\} \).

An intuitive approach may be to calculate the detector values
\[
|V_k^{*\langle i \rangle}| = \left| \sum_{t=m+1}^{m+k} \left[ \hat{D}_m^{*\langle i \rangle} \right]^{-1/2} \hat{p}^{\langle i \rangle} T (\hat{\theta}_m) \right|
\]
from each sample \( Y^{*\langle i \rangle} \) with \( \hat{\theta}_m \) the QLL contributions and
\[
\hat{D}_m^{*\langle i \rangle} = \frac{1}{m} \sum_{t=1}^{m} \hat{p}^{\langle i \rangle} T (\hat{\theta}_m) \left[ \hat{p}^{\langle i \rangle} T (\hat{\theta}_m) \right]^T
\]
the estimate of the matrix \( D \) from Assumption 4.2 based on the first \( m \) observations of \( Y^{*\langle i \rangle} \). But since we are not interested in using the exact limit distribution of the detector, the matrix \( D \) is substituted by the identity matrix to avoid the additional uncertainty that goes along with the
matrix estimation. Further simulations that are not part of this article showed that this approach yields a remarkable decrease of the size distortions compared to the use of an estimate of $D$. Of course, other choices can be used instead of the identity matrix such a matrix more close to $D$. One possibility is to use an estimate of the diagonal of $D$. In any case, this would mean having to perform an estimation process that is precisely what we try to avoid.

Denote the resulting detector as $|\hat{V}_k^*(i)|$ and the maximum of the scaled detector values gained from sample $Y^*(i)$ as

$$T^*(i) := \max_{1 \leq k \leq \lfloor mB \rfloor} \frac{|V_k^*(i)|}{m^{1/2} \left(1 + \frac{k}{m}\right) b\left(\frac{k}{m}\right)} \text{, for } i \in \{1, \ldots, b_{BT}\}.$$ 

The $(1 - \tilde{\alpha})$ quantile of $\{T^*(1), \ldots, T^*(b_{BT})\}$ can be used as a critical value in finite sample applications. Although a detailed analysis of these critical values and their properties lies beyond the scope of this article, they show a satisfying behavior in simulations.

Lastly, we investigate the asymptotic distribution of the detector under a parameter change. Recall that under the alternative of a structural break at an unknown position in the monitoring period, the parameter vector changes from $\theta$ to $\theta^*$ at the $k^*$-th point of the monitoring period. To obtain asymptotic properties, some additional assumptions are imposed, which are counterparts to the Assumptions 3.3, 3.4 and 3.5 in the post break period. Since we assume, that the change only affects the parameters but not the underlying structure of the process, we can expect that Assumption 3.6 is still valid after the change.

**Assumption 4.4.** $\theta^* \in U$.

**Assumption 4.5.**

$$\max_{1 \leq i \leq p} \alpha_i^* \quad \text{and} \quad \max_{1 \leq i \leq p} \beta_i^* < 1 \quad \text{and} \quad |\beta^*| < 1$$

**Assumption 4.6.**

$$E \left[ \ln \left( [\beta^*]^2 + [\alpha^*]^2 \frac{4(2p + 1)\sqrt{p}}{\sqrt{C_N}C_q^* \|\eta_t\|_2^2} \right) \right] < 0$$
with constants $C' = \frac{\lambda_{\min}((1 - \alpha^* - \beta^*)\tilde{Q}^*)}{1 - [\beta^*]^2}$ and $C_q' = \frac{(1 - \alpha^* - \beta^*)\min_{1 \leq i \leq p} q_i^2}{1 - [\beta^*]^2}$

**Assumption 4.7.** For $\|\cdot\|$, the norm from Assumption 3.5, we have:

$$E(\|\eta\|^8) < \infty \quad \text{and} \quad \lambda_{\max} \left( \sup_{x \in \mathbb{R}^d} E \left[ \|\otimes_T T_i^* \| \mid X_{t-1} = x \right] \right) < 1,$$

where $\otimes_k$ is the $k$-fold Kronecker product of identical matrices and $T_i^*$ the counterpart of $T_i$ in (3.2), that depends on $\theta^*$ instead of $\theta$.

**Theorem 4.2.** Under the alternative of a structural break and Assumptions 2.1, 3.1, 3.2, 3.6, 3.7, 3.8, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, we have

$$\lim_{m \to \infty} P_{H_1} \{ \tau_m < \infty \} = 1.$$

Since it takes some time until the influence of the post break observations on the detector is strong enough to report the presence of a changepoint, it has to be assumed that in general the changepoint location is not consistent with the first hitting time $\tau_m$. Once the sheer presence of a change in the parameter vector is indicated, the position of the changepoint has to be estimated. One possible method works analogously to the estimators in Wied et al. (2012) and in Wied and Galeano (2013). It is defined as

$$\hat{k} := \arg \max_{1 \leq k \leq \tau_m - 1} \frac{k}{\sqrt{\tau_m}} \left| \frac{1}{\tau_m - 1} \sum_{t=m+1}^{m+\tau_m - 1} \hat{l}_t \left( \hat{\theta}_m \right) - \frac{1}{\hat{k}} \sum_{t=m+1}^{m+k} \hat{l}_t \left( \hat{\theta}_m \right) \right|. \quad (4.2)$$

Though a detailed analysis of its properties lies beyond the scope of this paper, estimators of this type showed satisfactory behavior in simulations and applications. That is why we use (4.2) to estimate the location of the changepoint throughout the next sections.
5 Simulations

This section is devoted to the investigation of the procedure’s performance under various simulation settings. Under the null as well as under alternative hypotheses, some parameters have to be specified. First, we choose the length of the historical period as \( m \in \{500,1,000,2,000\} \). In terms of trading days, this roughly equals 2, 4 and 8 years, respectively. We assume that the length of the monitoring period is considerably smaller than the length of the historical period with \( B \in \{0.1,0.2,\ldots,0.5\} \). The dimension of the random vectors is \( p \in \{3,5\} \) and the tuning parameter is chosen as \( \gamma \in \{0,0.2,0.4\} \). These values support the ability of the monitoring procedure to detect early or later appearing structural breaks. In any case, we simulated 1,000 time series and applied our procedure to them. Note that all of the simulations are carried out for a significance level of \( \tilde{\alpha} = 0.05 \) and for the parametric Bootstrap type procedure. The innovations of the DCC models follow a multivariate standard normal distribution.

5.1 Simulations Under the Null Hypothesis

First, we investigate the performance of the monitoring scheme under the null hypothesis of no structural break in the parameter vector. For each vector component, the variance parameters are chosen either as \( \phi_i = (0.01,0.05,0.9)' \) or as \( \phi_i = (0.01,0.2,0.7)' \), for all \( i \in \{1,\ldots,p\} \). Therefore, the second case indicates a stronger effect of single shocks on the volatility of future observations. The correlation structure is determined by the parameters \( (\alpha,\beta) = (0.05,0.9) \) and the constant unconditional correlation matrix \( \bar{Q}_p \), which is defined as

\[
\bar{Q}_3 = \begin{bmatrix} 1 & 0.5 & 0.1 \\ 0.5 & 1 & 0.5 \\ 0.1 & 0.5 & 1 \end{bmatrix} \quad \text{and} \quad \bar{Q}_5 = \begin{bmatrix} 1 & 0.5 & 0.3 & 0.2 & 0.1 \\ 0.5 & 1 & 0.5 & 0.3 & 0.2 \\ 0.3 & 0.5 & 1 & 0.5 & 0.3 \\ 0.2 & 0.3 & 0.5 & 1 & 0.5 \\ 0.1 & 0.2 & 0.3 & 0.5 & 1 \end{bmatrix}.
\]

The results in Table 1 suggest that the empirical size increases with \( B \), which is plausible since
larger values of this parameter imply a growing length of the monitoring period and thus more uncertainty. While larger values of \( m \) and \( \gamma \) reduce the size distortions, higher dimensions tend to increase the probability to commit a type I error. Especially the influence of variations in the length of the historical period and the dimension are as expected.

Furthermore, the empirical size is distinctly higher when the variance parameters are chosen as \( \phi_i = (0.01, 0.05, 0.9)' \). This result was to be expected since the sum \( \alpha_i + \beta_i \) is closer to one, i.e. we are closer to the case of a unit root process than in the second scenario.

<table>
<thead>
<tr>
<th>( m )</th>
<th>500</th>
<th>1,000</th>
<th>2,000</th>
<th>( m )</th>
<th>500</th>
<th>1,000</th>
<th>2,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_i = (0.01, 0.05, 0.9)' )</td>
<td></td>
<td></td>
<td></td>
<td>( \phi_i = (0.01, 0.2, 0.7)' )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( B = 0.1 )</td>
<td>( \gamma = 0 )</td>
<td>0.124</td>
<td>0.088</td>
<td>0.068</td>
<td>0.068</td>
<td>0.077</td>
<td>0.047</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.2 )</td>
<td>0.133</td>
<td>0.084</td>
<td>0.074</td>
<td>0.067</td>
<td>0.070</td>
<td>0.052</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.4 )</td>
<td>0.116</td>
<td>0.082</td>
<td>0.070</td>
<td>0.064</td>
<td>0.058</td>
<td>0.066</td>
</tr>
<tr>
<td>( B = 0.2 )</td>
<td>( \gamma = 0 )</td>
<td>0.150</td>
<td>0.101</td>
<td>0.094</td>
<td>0.069</td>
<td>0.057</td>
<td>0.088</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.2 )</td>
<td>0.151</td>
<td>0.086</td>
<td>0.091</td>
<td>0.069</td>
<td>0.077</td>
<td>0.087</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.4 )</td>
<td>0.118</td>
<td>0.080</td>
<td>0.083</td>
<td>0.065</td>
<td>0.060</td>
<td>0.060</td>
</tr>
<tr>
<td>( B = 0.3 )</td>
<td>( \gamma = 0 )</td>
<td>0.177</td>
<td>0.111</td>
<td>0.089</td>
<td>0.120</td>
<td>0.087</td>
<td>0.081</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.2 )</td>
<td>0.151</td>
<td>0.094</td>
<td>0.073</td>
<td>0.095</td>
<td>0.075</td>
<td>0.083</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.4 )</td>
<td>0.143</td>
<td>0.084</td>
<td>0.086</td>
<td>0.071</td>
<td>0.054</td>
<td>0.069</td>
</tr>
<tr>
<td>( B = 0.4 )</td>
<td>( \gamma = 0 )</td>
<td>0.213</td>
<td>0.113</td>
<td>0.118</td>
<td>0.106</td>
<td>0.109</td>
<td>0.104</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.2 )</td>
<td>0.193</td>
<td>0.120</td>
<td>0.105</td>
<td>0.110</td>
<td>0.098</td>
<td>0.106</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.4 )</td>
<td>0.147</td>
<td>0.094</td>
<td>0.072</td>
<td>0.077</td>
<td>0.066</td>
<td>0.063</td>
</tr>
<tr>
<td>( B = 0.5 )</td>
<td>( \gamma = 0 )</td>
<td>0.197</td>
<td>0.118</td>
<td>0.141</td>
<td>0.129</td>
<td>0.110</td>
<td>0.122</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.2 )</td>
<td>0.179</td>
<td>0.135</td>
<td>0.097</td>
<td>0.100</td>
<td>0.109</td>
<td>0.115</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.4 )</td>
<td>0.166</td>
<td>0.111</td>
<td>0.090</td>
<td>0.086</td>
<td>0.073</td>
<td>0.073</td>
</tr>
<tr>
<td>( B = 0.1 )</td>
<td>( \gamma = 0 )</td>
<td>0.139</td>
<td>0.093</td>
<td>0.079</td>
<td>0.080</td>
<td>0.085</td>
<td>0.062</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.2 )</td>
<td>0.141</td>
<td>0.104</td>
<td>0.067</td>
<td>0.071</td>
<td>0.066</td>
<td>0.047</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.4 )</td>
<td>0.117</td>
<td>0.093</td>
<td>0.072</td>
<td>0.070</td>
<td>0.048</td>
<td>0.060</td>
</tr>
<tr>
<td>( B = 0.2 )</td>
<td>( \gamma = 0 )</td>
<td>0.153</td>
<td>0.099</td>
<td>0.083</td>
<td>0.079</td>
<td>0.080</td>
<td>0.082</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.2 )</td>
<td>0.161</td>
<td>0.112</td>
<td>0.085</td>
<td>0.088</td>
<td>0.081</td>
<td>0.068</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.4 )</td>
<td>0.148</td>
<td>0.083</td>
<td>0.073</td>
<td>0.074</td>
<td>0.069</td>
<td>0.059</td>
</tr>
<tr>
<td>( B = 0.3 )</td>
<td>( \gamma = 0 )</td>
<td>0.181</td>
<td>0.109</td>
<td>0.087</td>
<td>0.102</td>
<td>0.118</td>
<td>0.116</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.2 )</td>
<td>0.161</td>
<td>0.110</td>
<td>0.098</td>
<td>0.087</td>
<td>0.098</td>
<td>0.090</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.4 )</td>
<td>0.148</td>
<td>0.087</td>
<td>0.085</td>
<td>0.074</td>
<td>0.064</td>
<td>0.071</td>
</tr>
<tr>
<td>( B = 0.4 )</td>
<td>( \gamma = 0 )</td>
<td>0.181</td>
<td>0.117</td>
<td>0.122</td>
<td>0.121</td>
<td>0.106</td>
<td>0.125</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.2 )</td>
<td>0.199</td>
<td>0.109</td>
<td>0.111</td>
<td>0.102</td>
<td>0.108</td>
<td>0.101</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.4 )</td>
<td>0.151</td>
<td>0.114</td>
<td>0.088</td>
<td>0.073</td>
<td>0.079</td>
<td>0.086</td>
</tr>
<tr>
<td>( B = 0.5 )</td>
<td>( \gamma = 0 )</td>
<td>0.198</td>
<td>0.111</td>
<td>0.131</td>
<td>0.152</td>
<td>0.126</td>
<td>0.140</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.2 )</td>
<td>0.186</td>
<td>0.111</td>
<td>0.123</td>
<td>0.122</td>
<td>0.120</td>
<td>0.118</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 0.4 )</td>
<td>0.182</td>
<td>0.107</td>
<td>0.107</td>
<td>0.092</td>
<td>0.084</td>
<td>0.059</td>
</tr>
</tbody>
</table>

Table 1: Empirical size for various parameter combinations.
5.2 Simulations Under Various Alternatives

In this section, we investigate the performance of the proposed procedure in view of different types of structural breaks. More precisely, we first simulate changes in the variance parameters followed by changes in the unconditional correlation matrix $\bar{Q}$.

Since the results under the null showed a strong dependency on the length of the monitoring period, the simulations under alternative scenarios will be limited to the case of monitoring periods with length $0.2m$. This choice of $B$ yields small deviations between the empirical and the theoretical size as the results from Table 1 suggest. As the length of the monitoring period depends on $m$, we define the location of the changepoint $k^*$ in terms of $m$ as $k^* = \lfloor mB\lambda^* \rfloor$, where $\lfloor \cdot \rfloor$ is the largest integer smaller than a given real number and the fraction $\lambda^*$ is chosen from $\{0.05, 0.3, 0.5\}$. This indicates changes located at the beginning or later in the monitoring period.

5.2.1 Changes in the Variance Parameters

We investigate two different settings of changes in the variance parameters. First, we assume that the vector $\phi_i = (0.01, 0.05, 0.9)'$ changes to $\phi_i^* = (0.005, 0.2, 0.7)'$ followed by a change from $\phi_i = (0.01, 0.2, 0.7)'$ to $\phi^* = (0.05, 0.05, 0.9)'$ for all $i \in \{1, \ldots, p\}$. These settings will be denoted as Scenario 1 and 2. Note that next to the actual variation in the parameters, which causes a change in the conditional variance structure, Scenario 1 implies a decrease in the unconditional variances of all components while Scenario 2 indicates a variance increase. The results for Scenario 1 can be seen in Tables 2 and 3 and those for Scenario 2 in Tables 4 and 5.

The power depends positively on the length of the historical period and negatively on the dimension of the random vectors. While the first result is as expected, the negative influence of $p$ on the power may be explained by the fact that the share of the $3p$ variance parameters in the group of all parameters decreases with growing dimension. Thus, changes in the variance parameters might be harder to detect if $p$ gets large.

The ability to detect parameter changes is distinctly higher for changes located at the beginning of the monitoring period than for later ones which is a typical property of sequential monitoring schemes based on the information of a historical sample, see for instance Berkes et al. (2004),
<table>
<thead>
<tr>
<th>$\lambda^*$</th>
<th>$m$</th>
<th>$\gamma$</th>
<th>Power</th>
<th>Empirical first hitting times</th>
<th>Location estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>500 ($k^* = 5$)</td>
<td>0</td>
<td>0.973</td>
<td>44.87 14.49 35 42 51</td>
<td>18.05 10.39 11 16 23</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.971</td>
<td>40.28 15.77 30 38 48</td>
<td>16.08 9.76 9 14 21</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.971</td>
<td>40.24 18.28 29 38 50</td>
<td>16.03 10.32 9 14 21</td>
</tr>
<tr>
<td></td>
<td>1.000 ($k^* = 10$)</td>
<td>0</td>
<td>1.000</td>
<td>60.40 15.46 51 57 68</td>
<td>24.40 11.71 17 22 29</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.999</td>
<td>52.88 16.09 42 50 60</td>
<td>21.18 10.53 14 19 26</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>1.000</td>
<td>47.09 19.83 36 45 56</td>
<td>19.20 11.89 12 17 24</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>500 ($k^* = 30$)</td>
<td>0</td>
<td>0.916</td>
<td>66.47 15.56 58 66 77</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.891</td>
<td>63.86 17.27 54 64 75</td>
<td>30.46 10.17 26 30 35</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.849</td>
<td>64.25 22.60 54 68 79</td>
<td>31 12.26 26 31 38</td>
</tr>
<tr>
<td></td>
<td>1.000 ($k^* = 60$)</td>
<td>0</td>
<td>0.998</td>
<td>110.78 20.19 97 109 122</td>
<td>60.59 12.06 55 60 66</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.998</td>
<td>107.62 22.10 94 106 119</td>
<td>59.84 13.31 55 60 66</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.994</td>
<td>107.05 28.90 94 108 123</td>
<td>58.73 17.64 54 60 66</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>500 ($k^* = 50$)</td>
<td>0</td>
<td>0.724</td>
<td>77.82 17.47 70.75 81 91</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.703</td>
<td>76.83 19.48 70 81 90</td>
<td>42.58 14.30 36 47 52</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.558</td>
<td>71.13 28.28 66 80 91</td>
<td>39.39 17.75 31 46 52</td>
</tr>
<tr>
<td></td>
<td>1.000 ($k^* = 100$)</td>
<td>0</td>
<td>0.965</td>
<td>148.56 22.61 135 149 163</td>
<td>92.07 20.31 87 97 103</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.972</td>
<td>148.36 24.02 136 149 162</td>
<td>93.03 19.56 88 97 103</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.931</td>
<td>150.02 36.68 140 156 172</td>
<td>92.43 24.40 90 98 104</td>
</tr>
</tbody>
</table>

Table 2: Power against changes in the parameters that imply a variance decrease (Scenario 1) and properties of the first hitting times $\tau_m$ and estimated changepoints $\hat{k}$ for $p = 3$.

<table>
<thead>
<tr>
<th>$\lambda^*$</th>
<th>$m$</th>
<th>$\gamma$</th>
<th>Power</th>
<th>Empirical first hitting times</th>
<th>Location estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>500 ($k^* = 5$)</td>
<td>0</td>
<td>0.927</td>
<td>44.46 13.82 36 42 50</td>
<td>17.23 9.07 11 15 22</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.913</td>
<td>39.51 15.13 30 37 45</td>
<td>15.31 8.99 9 13 19</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.892</td>
<td>38.88 16.83 29 37 48</td>
<td>15.46 10.01 9 13 20</td>
</tr>
<tr>
<td></td>
<td>1.000 ($k^* = 10$)</td>
<td>0</td>
<td>0.997</td>
<td>60.62 14.83 51 58 67</td>
<td>23.61 11.11 16 21 28</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.997</td>
<td>51.60 14.42 42 49 59</td>
<td>20.66 10.31 14 18 25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.996</td>
<td>47.04 18.90 37 45 57</td>
<td>18.85 11.01 12 17 24</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>500 ($k^* = 30$)</td>
<td>0</td>
<td>0.859</td>
<td>66.82 15.25 58 67 77</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.848</td>
<td>64.89 16.99 56 65 75</td>
<td>31.01 10.60 26 30 37</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.806</td>
<td>64.09 22.07 55 66 78</td>
<td>30.23 11.81 25.25 30 37</td>
</tr>
<tr>
<td></td>
<td>1.000 ($k^* = 60$)</td>
<td>0</td>
<td>0.993</td>
<td>108.95 17.86 97 107 119</td>
<td>59.93 12.89 54 60 65</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.992</td>
<td>106.35 20.53 94 106 118</td>
<td>58.72 13.69 54 59 64</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.992</td>
<td>107.68 28.43 97 109 123</td>
<td>58.45 16.38 54 60 66</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>500 ($k^* = 50$)</td>
<td>0</td>
<td>0.680</td>
<td>79.04 16.92 73 82 91</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.763</td>
<td>75.54 20.58 70 82 91</td>
<td>42.48 14.36 37 46 51</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.553</td>
<td>70.67 30.36 65 82 93</td>
<td>38.90 18.00 31 46 51</td>
</tr>
<tr>
<td></td>
<td>1.000 ($k^* = 100$)</td>
<td>0</td>
<td>0.973</td>
<td>147.58 22.32 136 148 161</td>
<td>91.81 19.18 86 96 102</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.972</td>
<td>145.77 24.53 135 148 161</td>
<td>90.26 21.93 84 97 102</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
<td>0.947</td>
<td>148.30 38.44 139 155 169</td>
<td>90.00 25.49 87 97 103</td>
</tr>
</tbody>
</table>

Table 3: Power against changes in the parameters that imply a variance decrease (Scenario 1) and properties of the first hitting times $\tau_m$ and estimated changepoints $\hat{k}$ for $p = 5$.

Wied and Galeano (2013) or Pape et al. (2016). Furthermore, parameter changes that lead to decreased unconditional variance can be detected much more reliably than changes that entail
### Appendix A: Table 4

<table>
<thead>
<tr>
<th>$\lambda^*$</th>
<th>$m$</th>
<th>$\gamma$</th>
<th>Power</th>
<th>Mean</th>
<th>SD</th>
<th>$Q_{0.25}$</th>
<th>$Q_{0.5}$</th>
<th>$Q_{0.75}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>500</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>($k^* = 5$)</td>
<td>0</td>
<td>0.768</td>
<td>64.97</td>
<td>19.92</td>
<td>51</td>
<td>66</td>
<td>80</td>
<td>24.62</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>0.702</td>
<td>56.90</td>
<td>23.55</td>
<td>39.25</td>
<td>57</td>
<td>75</td>
<td>21.57</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>0.558</td>
<td>49.34</td>
<td>29.99</td>
<td>21.25</td>
<td>48</td>
<td>75.75</td>
<td>20.44</td>
</tr>
<tr>
<td><strong>500</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>($k^* = 10$)</td>
<td>0</td>
<td>0.974</td>
<td>116.05</td>
<td>32.61</td>
<td>95</td>
<td>114</td>
<td>138</td>
<td>39.92</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>0.955</td>
<td>107.65</td>
<td>38.44</td>
<td>80</td>
<td>107</td>
<td>133</td>
<td>36.77</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>0.810</td>
<td>107.81</td>
<td>49.94</td>
<td>72</td>
<td>109.5</td>
<td>146</td>
<td>39.06</td>
</tr>
<tr>
<td><strong>500</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>($k^* = 30$)</td>
<td>0</td>
<td>0.527</td>
<td>76.44</td>
<td>15.92</td>
<td>65</td>
<td>79</td>
<td>90</td>
<td>32.30</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>0.482</td>
<td>73.28</td>
<td>18.45</td>
<td>61</td>
<td>76</td>
<td>88</td>
<td>31.38</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>0.271</td>
<td>62.76</td>
<td>27.40</td>
<td>46</td>
<td>68</td>
<td>85</td>
<td>29.00</td>
</tr>
<tr>
<td><strong>500</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>($k^* = 50$)</td>
<td>0</td>
<td>0.809</td>
<td>151.87</td>
<td>28.70</td>
<td>130</td>
<td>155</td>
<td>176</td>
<td>58.77</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>0.727</td>
<td>145.37</td>
<td>33.39</td>
<td>121</td>
<td>149</td>
<td>172</td>
<td>55.86</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>0.457</td>
<td>142.46</td>
<td>47.80</td>
<td>122</td>
<td>153</td>
<td>179</td>
<td>54.78</td>
</tr>
<tr>
<td><strong>500</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>($k^* = 100$)</td>
<td>0</td>
<td>0.505</td>
<td>167.32</td>
<td>24.59</td>
<td>153</td>
<td>172</td>
<td>187</td>
<td>86.40</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>0.445</td>
<td>161.34</td>
<td>32.89</td>
<td>146</td>
<td>168</td>
<td>184</td>
<td>83.36</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>0.221</td>
<td>138.81</td>
<td>61.53</td>
<td>122</td>
<td>164</td>
<td>182</td>
<td>74.26</td>
</tr>
</tbody>
</table>

### Table 4: Power against changes in the parameters that imply a variance increase (Scenario 2) and properties of the first hitting times $\tau_m$ and estimated changepoints $k$ for $p = 3$.  

### Table 5: Power against changes in the parameters that imply a variance increase (Scenario 2) and properties of the first hitting times $\tau_m$ and estimated changepoints $k$ for $p = 5$.  

---

**Table 5**: Power against changes in the parameters that imply a variance increase (Scenario 2) and properties of the first hitting times $\tau_m$ and estimated changepoints $k$ for $p = 5$.  

---

This property underlines the results under the null, which suggest a stronger tendency of the detector to cross the threshold function if the initial variance parameters are chosen.
as $\phi_i = (0.01, 0.05, 0.9)'$ rather than $\phi_i = (0.01, 0.2, 0.7)'$.

In addition and consistently with the results under the null, the power decreases with the tuning parameter $\gamma$. This effect occurs even if the structural break is located at a later point in the monitoring period and should be detected more frequently if a higher value of $\gamma$ is used.

The results concerning the estimated changepoint locations in Tables 2-5 suggest that the position of changepoints located at a fraction of $\lambda^* = 0.3$ of the monitoring period, can be estimated without large distortions while earlier and later changes, respectively, are systematically placed too early and late, respectively, in the dataset. Note that the results for the estimated changepoint locations depend strongly on the properties of the first hitting times since these define the length of the subsample, that is used to locate the changepoint.

### 5.2.2 Changes in the Correlation Parameters

Another possible alternative scenario is a change in the correlation structure. We assume that the variance parameters as well as $\alpha$ and $\beta$ stay constant, whereas the matrix $\bar{Q}$ changes from $I_p$ to $\bar{Q}^*$. The latter one is a matrix whose main diagonal entries are equal to one, while all of the diagonal entries are $\Delta$ with $\Delta \in \{0.1, \ldots, 0.9\}$. The variance parameters and $\alpha$ and $\beta$ are chosen as in Section 5.1.

The power results for changes at fraction $\lambda^* = 0.05$ of the monitoring period are illustrated in Figure 1 for simulated time series of dimension 3 or 5, a historical period consisting of 1.000 data points and tuning parameter $\gamma = 0.2$. The results reveal problems to detect correlation changes of moderate magnitude for both choices of the vector of variance parameters. However, the power curve has a large slope for higher values and converges to one. While smaller changes in the correlation parameters can be detected more frequently if the variance parameters are chosen as $\phi_i = (0.01, 0.05, 0.9)'$ rather than $\phi_i = (0.01, 0.2, 0.7)'$, the opposite is true for larger values of $\Delta$.

The fact that some of the power results are quite low, suggests that the QLL function seems to be very flat in some regions, such that some kinds of parameter changes are hard to detect.
6 Empirical Results

To investigate the performance of the proposed monitoring scheme under real conditions, the procedure is applied to a group of asset returns. Due to the fact that a conjoint modeling seems to be appropriate for the returns of assets from the same industrial sector and monetary area, we choose the assets of several insurance companies, which are listed at different stock exchanges throughout Europe. More precisely, we monitor the log returns of the assets of Allianz (abbreviated by All), AXA, Generali (Gen), ING and Munich Re (MRe) in the time from 1998-05-11 to 2016-10-25. Engle (2002) argued, that the DCC model is in principle well-suited to model the typical features of multivariate return time series. Furthermore, Bollerslev (1986) pointed out, that even GARCH models of order (1,1) are capable of explaining the behavior of log returns very well. Thus, we use GARCH(1,1) models for the univariate conditional variance equations (2.4), which is in line with our approach in Section 2.

As the results in Table 1 suggest, that the size increases considerably with the length of the monitoring period and hence with $B$, we monitor the data by the use of a stepwise approach, which
works as follows.

(1) Use the first \( m \) observations as historical period and monitor the following \( Bm \) observations for a parameter change.

(2a) If a changepoint is detected in the subsample, estimate the location of the changepoint and cut off all of the pre change observations. Then, restock the subsample to \( m \) observations and return to step (1). If there are not enough observations left to restock the historical sample to \( m \) observations or the monitoring dataset to \( mB \) observations, terminate the procedure.

(2b) If no changepoint is detected in the subsample, cut off the first \( mB \) data points of the historical sample and add the data of the previous monitoring period to the historical dataset. Return to step (1). If there are not enough observations left to restock the historical sample to \( m \) observations or the monitoring dataset to \( mB \) observations, terminate the procedure.

The results for \( \gamma = 0.2, B = 0.2 \) and different lengths of the historical period can be seen in Table 6. The estimated changepoint locations for \( m = 1,000 \) are shown in Figure 2 with two of the monitored time series.

<table>
<thead>
<tr>
<th>( \gamma = 0.2, B = 0.2, m = 500 )</th>
<th>( \gamma = 0.2, B = 0.2, m = 1,000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_m )</td>
<td>( k )</td>
</tr>
<tr>
<td>2000/06/09</td>
<td>2000/05/22</td>
</tr>
<tr>
<td>2002/10/15</td>
<td>2002/10/10</td>
</tr>
<tr>
<td>2006/03/01</td>
<td>2005/12/05</td>
</tr>
<tr>
<td>2008/05/28</td>
<td>2008/05/22</td>
</tr>
<tr>
<td>2010/08/04</td>
<td>2010/06/09</td>
</tr>
<tr>
<td>2012/07/26</td>
<td>2012/07/05</td>
</tr>
<tr>
<td>2014/07/09</td>
<td>2014/07/07</td>
</tr>
</tbody>
</table>

Table 6: First hitting times and estimated changepoint locations.

Figure 2 shows that the time series of asset returns are split up into calm and more turbulent phases. The estimated changepoint locations in Table 6 can be linked to important economic events of the last two decades. The changepoint in 2002 marks the end of the crisis caused by the bursting of
the dotcom bubble and the start of a calmer period that was interrupted in 2008 by the financial crisis followed by the debt crisis. The last changepoint might indicate the beginning of a recovery phase of the stock markets.

![Graph of log returns for Allianz and Generali assets with detected changepoints for m = 1.000 from Table 6 (dashed gray lines).](image)

Estimates of the model parameters calculated from the data between two successive changepoints can be seen in Table 7. To measure the magnitude of the changes in the estimated parameters, the table also contains the Euclidian norm of the parameter vectors estimated from the subsamples as well as the Euclidian norm of the estimated vectors of variance and correlation parameters, respectively. The largest change in the parameters in terms of the Euclidian norm can be found between the period before the financial crisis and the period of the crisis itself. A large part of this phenomenon seems to be caused by the fact that the correlation of asset returns tends to increase in times of crisis, see Sandoval Jr. and De Paula Franca (2012) among others.
Table 7: Model parameters estimated from the data between successive detected changepoints for \( m = 1.000 \) from Table 6 and the euclidical norm of the estimated parameter vectors and the estimated vectors of variance and correlation parameters.

### Conclusion

We present a method to detect changes in the parameter vector of the DCC model proposed by Engle (2002) which is based on quasi-log-likelihood scores and allows to detect changes in the conditional and unconditional variance and covariance structure. We analyze the size and power properties of the presented procedure and apply it to a group of log returns that belong to the assets of several insurance companies. In applications it turns out as a difficult problem that the assumption of a historical period which is free from structural breaks cannot be checked with
a known retrospective method. The search for a solution for this problem is left as a task for future research. Also, the statistic used is designed for situations in which many model parameters change. Thus, this statistic may not useful for detecting changes in one or few model parameters. Even if such changes are detected it would be complicated to distinguish which model parameters have changed already. The development of statistics adapted to these situations is also left for future research. Finally, note that we have focus on detecting the presence of changepoints. A complete development and understanding of the implications of these changepoints in issues such as forecasting or the estimation of risk measures deserve their own space.

References


