Supplementary Material to: Sequential Detection of Parameter Changes in Dynamic Conditional Correlation Models

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Contents

1 The Partial Derivatives of the DCC QLL function ................................................. 2
   1.1 Notation and Transformation Matrices ......................................................... 2
   1.2 Some Calculation Rules for Matrices ......................................................... 3
   1.3 The First Order Partial Derivatives with Respect to the Variance Parameters ..... 4
   1.4 The First Order Partial Derivatives with Respect to the Correlation Parameters .. 5
   1.5 The Second Order Partial Derivatives with Respect to $\theta$ ............................ 6

2 The Proofs of the Theoretical Statements .......................................................... 10
   2.1 The Proof of Proposition 4.1 ................................................................. 10
   2.2 The Proof that the Second Order Partial Derivatives are of Finite Expectation .... 27
   2.3 The Proof of Theorem 4.1 ................................................................. 34
   2.4 The Proof of Theorem 4.2 ................................................................. 36

3 The Proofs of the Calculation Rules in Section 1.2 ............................................ 38
   3.1 The Proof of Calculation Rule 1 ......................................................... 38
   3.2 The Proof of Calculation Rule 4 ......................................................... 38
1. The Partial Derivatives of the DCC QLL Function

1.1 Notation and Transformation Matrices

Recall that the quasi log likelihood (QLL) function of the DCC model introduced in Section 2.2 of the paper is given as:

\[
L_T(\theta) = \sum_{t=1}^{T} l_t(\theta) \text{ with }
\]

\[
l_t(\theta) = -\frac{1}{2} \left( p \cdot \log 2\pi + \log \det (H_t) + y_t'H_t^{-1}y_t \right) = -\frac{1}{2} \left( p \cdot \log 2\pi + 2 \log \det (D_t) + \log \det (R_t) + z_t'R_t^{-1}z_t \right)
\]

As argued in Hafner and Herwatz (2008), it is sufficient to look at the lower diagonal entries of \( R_t \) in detail since the latter one is a symmetric matrix with ones on the main diagonal. Throughout the next sections, several matrices are used to interchange the position of the entries in vectors and matrices, see for instance Lütkepohl (1996) or Hafner and Herwatz (2008):

- **vec(\cdot)**: the vec operator that stacks the entries of a matrix into a vector.
- **vech(\cdot)**: the vech operator that stacks the diagonal and lower diagonal entries of a symmetric matrix into a vector.
- **vecl(\cdot)**: the vecl operator that stacks the lower diagonal entries of a symmetric matrix into a vector.
- **\( K_{mn} \)**, the commutation matrix: \( \text{vec} (A') = K_{mn} \cdot \text{vec} (A) \) for a \((m \times n)\) matrix \( A \).
- **\( D_p \)**, the duplication matrix: \( D_p \cdot \text{vech}(A) = \text{vec}(A) \) for a symmetric \((p \times p)\) matrix \( A \).
- **\( D_p^+ \)**, the Moore Penrose inverse of \( D_p \). In general this is not the elimination matrix \( L_p \) with \( L_p \cdot \text{vec}(A) = \text{vech}(A) \), which is some generalized inverse.
- **\( D_{p,-} \)**, the matrix that results after deleting those columns from \( D_p \) that refer to the main diagonal entries of a symmetric \((p \times p)\) matrix \( A \) when \( D_p \) is multiplied by \( \text{vech}(A) \).
- **\( D_{p,-}^+ \)**, the Moore Penrose inverse of \( D_{p,-} \). This matrix is obtained when those rows that refer to the main diagonal elements of a symmetric \((p \times p)\) matrix \( A \) in vector \( \text{vech}(A) \) are deleted from \( D_p^+ \). Note, that for a symmetric \((p \times p)\) matrix \( A \), we have \( \text{vecl}(A) = D_{p,-}^+ \cdot \text{vec}(A) \).
1. THE PARTIAL DERIVATIVES OF THE DCC QLL FUNCTION

Furthermore, the following notation is used throughout the Supplementary Material:

- Applied to vectors or matrices, $|\cdot|$ is a norm that gives the modulus of the largest absolute entry.

- The number of lower diagonal elements of a $(p \times p)$ matrix is denoted by $p^+ = \frac{1}{2}(p + 1)p$ if the main diagonal entries are included and by $p^- = \frac{1}{2}(p - 1)p$ if they are excluded.

- $0_n$ is an $n$ dimensional vector of zeros and $0_{m \times n}$ an $(m \times n)$ matrix of zeros. Furthermore, $I_p$ denotes the $p$ dimensional identity matrix.

- For an $(m \times n)$ matrix $X$, $[X]_{ij}$ is the entry that is located at position $(i, j)$.

1.2 Some Calculation Rules for Matrices

The following calculation rules will be needed throughout the next sections:

**CR1** For the transformation matrices from Section 1.1, we have

$$D^+_{p,-} = \frac{1}{2}(D_{p,-})'.$$

The proof of this statement can be found in Section 3.

**CR2** Lütkepohl (1996), 10.4.2(1): $X \sim (m, m)$ symmetric:

$$\frac{\partial \text{vec}(X)}{\partial \text{vech}(X)'}, \quad \frac{\partial \text{vec}(X)}{\partial \text{vecl}(X)'} = D_m, \quad \frac{\partial \text{vec}(X)}{\partial \text{vech}(X)'} = D_{m,-} \overset{\text{CR1}}{=} 2(D_{m,-}')'. $$

**CR3** For symmetric matrices $X \sim (m, m)$ and $Y(X) \sim (n, n)$, we have

$$\frac{\partial \text{vech}(Y(X))}{\partial \text{vecl}(X)'} = \frac{\partial \text{vec}(Y(X))}{\partial \text{vec}(X)'}, \quad \frac{\partial \text{vec}(Y(X))}{\partial \text{vecl}(X)'} = L_n \frac{\partial \text{vec}(Y(X))}{\partial \text{vecl}(X)'} D_{m,-}.$$ 

This is a direct consequence of CR2.

**CR4** For symmetric $(n \times n)$ matrices $X$ and $Y(X)$, we have

$$\frac{\partial \text{vec}(XYX)}{\partial \text{vec}(X)'} = (X \otimes X) \frac{\partial \text{vec}(Y)}{\partial \text{vec}(X)'} + (XY \otimes I_n + I_n \otimes XY).$$

The proof of this statement can be found in Section 3.
1.3 The First Order Partial Derivatives with Respect to the Variance Parameters

Since the conditional correlation matrix $R_t$ does not depend on the variance parameters, the partial derivatives of the QLL contributions with respect to $\theta_1$ are similar to (3.2) in Nakatani and Teräsvirta (2009):

$$\frac{\partial l_t(\theta)}{\partial \theta_1} = -\frac{\partial \log \det D_t}{\partial \theta_1} = -\frac{1}{2} \frac{\partial \mathbf{y}_t H^{-1}_t \mathbf{y}_t}{\partial \theta_1} = -\frac{1}{2} \frac{\partial \text{vec}(D_t)^\prime}{\partial \theta_1} \text{vec}\left(2D_t^{-1} - D_t^{-1}R_t^{-1}z t z_t^\prime - z_t R_t^{-1}D_t^{-1}\right)$$

The partial derivatives of the non zero diagonal entries of $D_t$ with respect to $\phi_i = (\omega_i, \alpha_i, \beta_i)'$ are given as

$$g_{it} := \frac{\partial h_{it}^{1/2}}{\partial \phi_i} = \frac{1}{2} h_{it}^{-1/2} \left(v_{jt-1} + \beta_j \frac{\partial h_{jt-1}}{\partial \phi_j}\right)$$

and

$$\frac{\partial \text{vec}(D_t)^\prime}{\partial \theta_1} = \begin{bmatrix}
g_{1t} & 0_{3 \times p} & 0_3 & \ldots & 0_3 & 0_{3 \times p} & 0_3 \\
0_3 & 0_{3 \times p} & g_{2t} & 0_{3 \times p} & \ldots & 0_3 & 0_{3 \times p} & 0_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0_3 & 0_{3 \times p} & 0_3 & 0_{3 \times p} & \ldots & g_{p-1,t} & 0_{3 \times p} & 0_3 \\
0_3 & 0_{3 \times p} & 0_3 & 0_{3 \times p} & \ldots & 0_3 & 0_{3 \times p} & g_{pt}
\end{bmatrix}.$$

Note that for theoretical considerations we use representations that depend on an infinite past of observations $y_t$, while for simulation or parameter estimation we use the recursive form based on starting values for time $t = 0$.

$$\begin{align*}
\frac{\partial h_{it}}{\partial \omega_i} &= 1 + \beta_i \frac{\partial h_{it-1}}{\partial \omega_i} = \frac{1}{1 - \beta_i} \\
\frac{\partial h_{it}}{\partial \alpha_i} &= \gamma_{it-1}^2 + \beta_i \frac{\partial h_{it-1}}{\partial \alpha_i} = \sum_{n=0}^{\infty} \beta_n \gamma_{i,t-n}^2 \\
\frac{\partial h_{it}}{\partial \beta_i} &= h_{it-1} + \beta_i \frac{\partial h_{it-1}}{\partial \beta_i} = \sum_{n=0}^{\infty} \beta_n h_{i,t-n-1} = \frac{\omega_i}{1 - \beta_i^3} + \alpha_i \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \beta_n \gamma_{i,t-n-k-2}^2
\end{align*}$$

Starting values for estimation or simulation can be chosen as in the ccgarch package in R where $h_0 = (h_{10}, \ldots, h_{p0})'$ is chosen as $(s_1^2, \ldots, s_p^2)'$ with $s_j^2 = \frac{1}{T} \sum_{t=1}^{T} y_{jt}^2$, $j = 1, \ldots, p$. Hence, $v_{0} = (1, s_1^2, s_2^2)'$, $j = 1, \ldots, p$ and $(\frac{\partial h_{10}}{\partial \omega_1}, \ldots, \frac{\partial h_{p0}}{\partial \omega_p})' = 0_{p \times 3}$. 
1.4 The First Order Partial Derivatives with Respect to the Correlation Parameters

In this section, we view in detail the partial derivatives of the QLL contributions with respect to those parameters that determine the correlation structure of the DCC model. Similar to (3.3) in Nakatani and Teräsvirta (2009), we have

\[
\frac{\partial l(\theta)}{\partial \theta_2} = -\frac{1}{2} \frac{\partial \log \det R_t}{\partial \theta_2} - \frac{1}{2} \frac{\partial y_t' D_t^{-1} R_t^{-1} D_t^{-1} y_t}{\partial \theta_2} = -\frac{1}{2} \frac{\partial \text{vec}(R_t)'}{\partial \theta_2} \frac{\partial \text{vec}(R_t)'}{\partial \theta_2} \frac{\partial \log \det R_t}{\partial \theta_2}
\]

\[
= -\frac{1}{2} \frac{\partial \text{vec}(R_t)'}{\partial \theta_2} \frac{\partial \text{vec}(R_t)'}{\partial \theta_2} 2D_{p,-} \text{vec}(R_t^{-1}) - \frac{1}{2} \frac{\partial \text{vec}(R_t)'}{\partial \theta_2} 2D_{p,-} (-1) (R_t^{-1} \otimes R_t^{-1}) (D_t^{-1} \otimes D_t^{-1}) \text{vec}(y_t y_t')
\]

\[
= -\frac{\partial \text{vec}(R_t)'}{\partial \theta_2} D_{p,-} \left[ \text{vec}(R_t^{-1}) - (R_t^{-1} \otimes R_t^{-1}) (D_t^{-1} \otimes D_t^{-1})(y_t \otimes y_t) \right] = -\frac{\partial \text{vec}(R_t)'}{\partial \theta_2} \text{vec}\left( R_t^{-1} [I_p - z_{it} R_t^{-1}] \right)
\]

The second equality results from an extensive use of the chain rule, see 10.2.1(6) in Lütkepohl (1996). The third one uses 10.3.1(8), 10.3.3(10), 10.4.1(3), 10.4.2(1) and 10.6(1) in Lütkepohl (1996) and CR1 from Section 1.2. The remaining equalities work with CR2 and simple matrix computations.

In the following, we consider the partial derivatives of the lower diagonal entries of \( R_t \) with respect to the correlation parameters in detail. According to Hafner and Herwatz (2008), we have

\[
\frac{\partial \text{vec}(R_t)}{\partial \theta'_2} = D_{p,-} \frac{\partial \text{vec}(R_t)}{\partial \theta'_2} = D_{p,-} \frac{\partial \text{vec}(R_t)}{\partial \text{vech}(Q_t)} \frac{\partial \text{vech}(Q_t)}{\partial \theta'_2}
\]

where

\[
\frac{\partial \text{vec}(R_t)}{\partial \text{vech}(Q_t)} = \frac{\partial \text{vec}(R_t)}{\partial \text{vec}(Q_t')} \frac{\partial \text{vec}(Q_t')} {\partial \text{vech}(Q_t')} \frac{\partial \text{vech}(Q_t')}{\partial \text{vec}(Q_t')} \frac{\partial \text{vec}(Q_t')}{\partial \text{vech}(Q_t)} \frac{\partial \text{vech}(Q_t)}{\partial \text{vec}(Q_t')} \frac{\partial \text{vec}(Q_t)}{\partial \text{vech}(Q_t)} = \frac{\partial \text{vec}(Q_t')}{\partial \text{vech}(Q_t)} D_{p,-} \frac{\partial \text{vech}(Q_t)}{\partial \text{vech}(Q_t')}
\]

With a slight difference to Section 4.3 in Hafner and Herwatz (2008), the derivative of \( \text{vec}(Q_t' Q_t Q_t') \) with respect to \( \text{vec}(Q_t') \) is given as

\[
\frac{\partial \text{vec}(Q_t' Q_t Q_t')}{\partial \text{vec}(Q_t')} = \frac{\partial \text{vec}(Q_t')}{\partial \text{vech}(Q_t')} \text{vec}(Q_t) \frac{\partial \text{vech}(Q_t)}{\partial \text{vec}(Q_t')} = \frac{\partial \text{vec}(Q_t' Q_t Q_t')}{\partial \text{vec}(Q_t')} \text{vec}(Q_t) \frac{\partial \text{vech}(Q_t)}{\partial \text{vec}(Q_t')}
\]

and the derivative of \( \text{vech}(Q_t') \) with respect to \( \text{vech}(Q_t) \) as

\[
\frac{\partial \text{vech}(Q_t')}{\partial \text{vech}(Q_t)} = -\frac{1}{2} \text{diag} \left\{ \text{vech} \left( Q_t^{-3/2} \right) \right\} \cdot \text{diag} \left\{ \text{vech} \left( I_p \right) \right\}
\]

with \( q_{ij} = [Q_t]_{ij} \) and \( Q_t^{-3/2} := \left[ q_{ij}^{-3/2} \right]_{i,j=1,...,p} \).
Thus, (1.2) and (1.3) imply

\[
\frac{\partial \text{vecl}(R_i)}{\partial \theta_2} = D_{p,-}^\dagger \left[ (Q_i' \otimes Q_i') \frac{\partial \text{vecl}(Q_i)}{\partial \text{vecl}(Q_i)'} + (Q_i' Q_i \otimes I_p + I_p \otimes Q_i Q_i') \right] D_p \frac{\partial \text{vech}(Q_i^r)}{\partial \text{vech}(Q_i)'} \frac{\partial \text{vech}(Q_i^r)}{\partial \theta_2}.
\]

The derivative of \(\text{vech}(Q_i)\) with respect to \(\theta_2\) can be split up into

\[
\frac{\partial \text{vech}(Q_i)}{\partial \alpha} = -\text{vech}(\tilde{Q}) + \text{vech}(z_{l-1}z_{l-1}^\prime) + \beta \frac{\partial \text{vech}(Q_{l-1})}{\partial \beta} = -\frac{1}{1-\beta} \text{vech}(\tilde{Q}) + \sum_{n=0}^{\infty} \beta^n \text{vech}(z_{l-n-1}z_{l-n-1}^\prime)
\]

\[
\frac{\partial \text{vech}(Q_i)}{\partial \beta} = -\text{vech}(\tilde{Q}) + \text{vech}(Q_{l-1}) + \beta \frac{\partial \text{vech}(Q_{l-1})}{\partial \beta} = -\frac{1}{1-\beta} \text{vech}(\tilde{Q}) + \sum_{n=0}^{\infty} \beta^n \text{vech}(Q_{l-n-1})
\]

\[
\frac{\partial \text{vech}(Q_i)}{\partial \text{vecl}(\tilde{Q})} = (1 - \alpha - \beta) \frac{\partial \text{vech}(\tilde{Q})}{\partial \text{vecl}(\tilde{Q})'} + \alpha \frac{\partial \text{vech}(z_{l-1}z_{l-1}^\prime)}{\partial \text{vecl}(\tilde{Q})} + \beta \frac{\partial \text{vech}(Q_{l-1})}{\partial \text{vecl}(\tilde{Q})'} = \frac{1 - \alpha - \beta}{1 - \beta} L_p D_{p,-}
\]

As in the ccgarch package, we fix the initial values of \(\left(\frac{\partial \text{vech}(Q_0)}{\partial \alpha}, \frac{\partial \text{vech}(Q_0)}{\partial \beta}\right)\) as \((0, 0)\).

### 1.5 The Second Order Partial Derivatives with Respect to \(\theta\)

Along the lines of Nakatani and Teräsvirta (2009), the Hessian of the QLL contributions can be split up into several blocks:

\[
\frac{\partial^2 l_1(\theta)}{\partial \theta \partial \theta'} = \begin{bmatrix}
\frac{\partial^2 l_1(\theta)}{\partial \theta \partial \theta} & \frac{\partial^2 l_1(\theta)}{\partial \theta \partial \theta'} \\
\frac{\partial^2 l_1(\theta)}{\partial \theta' \partial \theta} & \frac{\partial^2 l_1(\theta)}{\partial \theta' \partial \theta'}
\end{bmatrix}
\]

#### 1.5.1 The Calculation of \(\frac{\partial^2 l_1(\theta)}{\partial \theta \partial \theta'}\)

Taking into consideration that the conditional correlation matrix \(R_i\) does not depend on the variance parameters, the upper left block of the Hessian is given analogously to (3.6) in Nakatani and Teräsvirta (2009):

\[
\frac{\partial^2 l_1(\theta)}{\partial \theta_1 \partial \theta_1'} = -\frac{1}{2} \frac{\partial}{\partial \theta_1} \left( \frac{\partial \text{vecl}(D_i)}{\partial \theta_1}' \text{vecl} \left( 2D_i^{-1} - D_i^{-1}R_i^{-1}z_i z_i' - z_i z_i' R_i^{-1}D_i^{-1} \right) \right)
\]

\[
= -\frac{1}{2} \frac{\partial \text{vecl}(D_i)}{\partial \theta_1}' \left[ -2 \left( D_i^{-1} \otimes D_i^{-1} \right) + \left( z_i z_i' \otimes D_i^{-1} R_i^{-1} D_i^{-1} \right) + \left( D_i^{-1} R_i^{-1} D_i^{-1} \otimes z_i z_i' \right) \\
+ \left( D_i^{-1} \otimes D_i^{-1} R_i^{-1} z_i z_i' \right) + \left( D_i^{-1} R_i^{-1} z_i z_i' \otimes D_i^{-1} \right) + \left( z_i z_i' R_i^{-1} D_i^{-1} \otimes D_i^{-1} \right) + \left( D_i^{-1} \otimes z_i z_i' R_i^{-1} D_i^{-1} \right) \right]
\]

\[
+ \frac{1}{2} \left( \text{vecl} \left( D_i^{-1} R_i^{-1} z_i z_i' \right) \otimes I_3p \right) + \frac{1}{2} \left( \text{vecl} \left( D_i^{-1} R_i^{-1} z_i z_i' \right) \otimes I_3p \right) - \left( \text{vecl} \left( D_i^{-1} \right) \otimes I_3p \right) \frac{\partial \text{vecl}(D_i)}{\partial \theta_1 \partial \theta_1'}.
\]
A closer look at the individual parts of the derivative yields

\[
\frac{\partial^2 \text{vech} (D_t)'}{\partial \theta_1 \partial \theta_1'} = \frac{\partial}{\partial \theta_1'} \text{vech} \left( \frac{\partial \text{vech} (D_t)'}{\partial \theta_1} D'_p \right) = \left( D_p \otimes I_{3p} \right) \frac{\partial}{\partial \theta_1'} \text{vech} \left( \frac{\partial \text{vech} (D_t)'}{\partial \theta_1} \right).
\]

Denote the non zero derivative blocks of the main diagonal entries of \(D\) as

\[
ge^{(2)}_{it} := \frac{\partial^2 h_{it}^{1/2}}{\partial \phi_i \partial \phi_i'}.
\]

Thus, the second order derivatives of \(\text{vech} (D_t)\) are given as

\[
\frac{\partial}{\partial \theta_1'} \text{vech} \left( \frac{\partial \text{vech} (D_t)'}{\partial \theta_1} \right) = \begin{bmatrix}
g^{(2)}_{1t} & 0_{3 \times 3 p^2} & 0_{3 \times 3} & 0_{3 \times 3(p-1)} & \cdots & 0_{3 \times 3} & 0_{3 \times 6p} & 0_{3 \times 3} \\
0_{3 \times 3} & g^{(2)}_{2t} & 0_{3 \times 3} & 0_{3 \times 3(p-1)} & \cdots & 0_{3 \times 3} & 0_{3 \times 6p} & 0_{3 \times 3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0_{3 \times 3} & 0_{3 \times 3} & g^{(2)}_{p-1,t} & 0_{3 \times 3} & \cdots & 0_{3 \times 3} & 0_{3 \times 6p} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3(p-1)} & \cdots & 0_{3 \times 3} & 0_{3 \times 6p} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3(p-1)} & \cdots & 0_{3 \times 3} & 0_{3 \times 6p}
g^{(2)}_{pt}
\end{bmatrix}.
\]

In detail, we have

\[
ge^{(2)}_{it} = -\frac{1}{4} h_{it}^{3/2} \frac{\partial^2 h_{it}}{\partial \phi_i^2} \left( V_{l,t-1} + \beta_l \frac{\partial h_{it-1}}{\partial \phi_i} \right) + \frac{1}{2} h_{it}^{3/2} \left( V_{l,t-1} + \beta_l \frac{\partial^2 h_{it-1}}{\partial \phi_i \partial \phi_i'} \right)
\]

with \(V_{lt} := \frac{\partial \nu_{it}}{\partial \phi_i} = \left( 0_3, 0_3, \frac{\partial h_{it}}{\partial \phi_i} \right) \) and

\[
\cdot \frac{\partial^2 h_{it}}{\partial \phi_i^2} = \beta_i \frac{\partial^2 h_{it-1}}{\partial \phi_i^2} = 0 \quad \cdot \frac{\partial^2 h_{it}}{\partial \phi_i \partial \phi_i'} = \beta_i \frac{\partial^2 h_{it-1}}{\partial \phi_i \partial \phi_i'} = 0 \quad \cdot \frac{\partial^2 h_{it}}{\partial \phi_i \partial \phi_i'} = \frac{\partial h_{it}}{\partial \phi_i} + \beta_i \frac{\partial^2 h_{it-1}}{\partial \phi_i \partial \phi_i'} = \frac{1}{(1-\beta_i)^2}
\]

\[
\cdot \frac{\partial^2 h_{it}}{\partial \phi_i^2} = \frac{\partial^2 h_{it-1}}{\partial \phi_i^2} = 0 \quad \cdot \frac{\partial^2 h_{it}}{\partial \phi_i \partial \phi_i'} = \frac{\partial h_{it}}{\partial \phi_i} + \beta_i \frac{\partial^2 h_{it-1}}{\partial \phi_i \partial \phi_i'} = \sum_{n=0}^{\infty} \beta_i \gamma_{l,t-1}^2
\]

\[
\cdot \frac{\partial^2 h_{it}}{\partial \phi_i^2} = 2 \frac{\partial h_{it-1}}{\partial \phi_i} + \beta_i \frac{\partial^2 h_{it-1}}{\partial \phi_i^2} = 2 \sum_{n=0}^{\infty} \beta_i \frac{\partial h_{it-n-1}}{\partial \phi_i} = \frac{2 \omega_i}{(1-\beta_i)^4} + 2 \alpha_i \sum_{k=0}^{\infty} \beta_i \sum_{l=0}^{\infty} \beta_i \gamma_{l,t-n-k-1}^2.
\]

Analogously to the approach in Section 1.3, a vector of zeros can be chosen as starting value for the recursive calculation of the second order partial derivatives of the conditional variances with respect to the variance parameters.
1.5.2 The Calculation of $\frac{\partial^2 l_i(\theta)}{\partial \theta_i \partial \theta'_2}$ and $\frac{\partial^2 l_i(\theta)}{\partial \theta_2 \partial \theta'_1}$

Analogously to (3.9) and (3.10) in Nakatani and Teräsvirta (2009) although limited to the partial derivatives of the lower diagonal entries of $R_t$ with respect to the correlation parameters, the off-diagonal blocks of the Hessian equal

$$\frac{\partial^2 l_i(\theta)}{\partial \theta_i \partial \theta'_2} = -\frac{1}{2} \frac{\partial}{\partial \theta_2} \left( \frac{\partial \text{vec} (D_i)^T}{\partial \theta_1} \text{vec} \left( 2D_t^{-1} - D_t^{-1}R_t^{-1}z_t'z_t'R_t^{-1}D_t^{-1} \right) \right)$$

$$\triangleq \frac{1}{2} \frac{\partial \text{vec} (D_i)^T}{\partial \theta_1} \left[ \frac{\partial \text{vec} \left( D_t^{-1}R_t^{-1}z_t'z_t' \right)}{\partial \theta'_2} + \frac{\partial \text{vec} \left( z_t'z_t'R_t^{-1}D_t^{-1} \right)}{\partial \theta'_2} \right]$$

$$\triangleq \frac{1}{2} \frac{\partial \text{vec} (D_i)^T}{\partial \theta_1} \left[ \left( z_t' \otimes D_t^{-1} \right) + \left( D_t^{-1} \otimes z_t' \right) \right] \frac{\partial \text{vec} (R_t^{-1})}{\partial \text{vec} (R_t)} \frac{\partial \text{vec} (R_t)}{\partial \text{vec} (R_t)} \frac{\partial \text{vec} (R_t)}{\partial \theta'_2}$$

$$\triangleq -\frac{\partial \text{vec} (D_i)^T}{\partial \theta_1} \left[ \left( z_t' \otimes R_t^{-1} \otimes R_t^{-1} \right) + \left( R_t^{-1} \otimes z_t' \otimes R_t^{-1} \right) \right] \left( \mathbb{I}_{p_t} \right) \frac{\partial \text{vec} (R_t)}{\partial \theta'_2}$$

$$\frac{\partial^2 l_i(\theta)}{\partial \theta_2 \partial \theta'_1} = \left[ \frac{\partial^2 l_i(\theta)}{\partial \theta_1 \partial \theta'_2} \right]' = -\frac{\partial \text{vec} (R_t)}{\partial \theta_2} \left[ \left( R_t^{-1}z_t'z_t'R_t^{-1} \right) \right] \frac{\partial \text{vec} (R_t)}{\partial \theta'_2}$$

1.5.3 The Calculation of $\frac{\partial^2 l_i(\theta)}{\partial \theta_2 \partial \theta'_2}$

Finally, the lower right block of the Hessian is given as

$$\frac{\partial^2 l_i(\theta)}{\partial \theta_2 \partial \theta'_2} = -\frac{\partial \text{vec} (R_t)}{\partial \theta_2} \left[ \left( R_t^{-1}z_t'z_t'R_t^{-1} \right) \right] \frac{\partial \text{vec} (R_t)}{\partial \theta'_2}$$

$$\triangleq -\left[ \left( \text{vec} (R_t^{-1}) \otimes \mathbb{I}_{p_t-2} \right) - \left( \text{vec} (R_t^{-1}z_t'z_t'R_t^{-1}) \otimes \mathbb{I}_{p_t-2} \right) \right] \frac{\partial \text{vec} (R_t)}{\partial \theta'_2}$$

$$\triangleq \frac{\partial \text{vec} (R_t)}{\partial \theta_2} \mathbb{D}_{p_t-2} \left[ \left( R_t^{-1} \otimes R_t^{-1} \right) - \left( R_t^{-1}z_t'z_t'R_t^{-1} \otimes R_t^{-1} \right) \right] \left( \mathbb{I}_{p_t} \right) \frac{\partial \text{vec} (R_t)}{\partial \theta'_2}$$

$$\triangleq \left[ \left( \text{vec} (R_t^{-1}) \otimes \mathbb{I}_{p_t-2} \right) + \left( \text{vec} (R_t^{-1}z_t'z_t'R_t^{-1}) \otimes \mathbb{I}_{p_t-2} \right) \right] \frac{\partial \text{vec} (R_t)}{\partial \theta'_2}$$

A closer look at the second order partial derivatives of the entries of the conditional correlation matrix with respect to the correlation parameters yields

$$\frac{\partial}{\partial \theta'_2} \text{vec} \left( \frac{\partial \text{vec} (R_t)}{\partial \theta_2} \right) = \frac{\partial}{\partial \theta'_2} \text{vec} \left( \frac{\partial \text{vec} \left( Q_t \right)'}{\partial \theta_2} \mathbb{D}_{p_t} \left( Q_t' \otimes Q_t' \right) \left( \mathbb{I}_{p_t-2} \right)' \right)$$

$$\triangleq \frac{\partial}{\partial \theta'_2} \text{vec} \left( \frac{\partial \text{vec} \left( Q_t \right)'}{\partial \theta_2} \frac{\partial \text{vec} (Q_t)' \text{vec} (Q_t)}{\partial Q_t} \mathbb{D}_{p_t} \left( Q_t' \otimes Q_t' \right) \left( \mathbb{I}_{p_t-2} \right)' \right).$$
By the use of 10.5.5(4) of Lütkepohl (1996) the first summand of (1.4) equals
\[
\left( D_{p,-}^+ \otimes \frac{\partial \vec{v}(Q_i)}{\partial \theta_2} D_p \right)^t \frac{\partial \vec{v}(Q_i) \otimes Q_i}{\partial \theta_2'} + \left( D_{p,-}^+ (Q_i \otimes Q_i) D_p \otimes I_{p^2} \right) \frac{\partial}{\partial \theta_2} \vec{v} \left( \frac{\partial \vec{v}(Q_i)}{\partial \theta_2} \right)
\]

Furthermore, 10.5.5(7) of Lütkepohl (1996) yields
\[
\frac{\partial \vec{v}(Q_i) \otimes Q_i}{\partial \theta_2'} = \left( I_p \otimes K_{pp} \otimes I_p \right) \left[ \frac{\partial \vec{v}(Q_i)}{\partial \theta_2'} \otimes \vec{v}(Q_i) + \vec{v}(Q_i) \otimes \frac{\partial \vec{v}(Q_i)}{\partial \theta_2'} \right]
\]
with \( \frac{\partial \vec{v}(Q_i)}{\partial \theta_2} = D_p \frac{\partial \vec{v}(Q_i)}{\partial \theta_2} \frac{\partial \vec{v}(Q_i)}{\partial \theta_2'} \).

Note that
\[
\frac{\partial}{\partial \theta_2} \vec{v} \left( \frac{\partial \vec{v}(Q_i)}{\partial \theta_2} \right) = K_{p^2} \frac{\partial}{\partial \theta_2} \vec{v} \left( \frac{\partial \vec{v}(Q_i)}{\partial \theta_2} \right)
\]
and that \( \frac{\partial}{\partial \theta_2} \vec{v} \left( \frac{\partial \vec{v}(Q_i)}{\partial \theta_2} \right) \) can be split up into several block matrices:

\[
\begin{bmatrix}
\frac{\partial^2 \vec{v}(Q_i)}{\partial \alpha^2} & \frac{\partial^2 \vec{v}(Q_i)}{\partial \alpha \beta} & \frac{\partial^2 \vec{v}(Q_i)}{\partial \beta^2} \\
\frac{\partial \vec{v}(Q_i)}{\partial \alpha} & \frac{\partial \vec{v}(Q_i)}{\partial \beta} & \frac{\partial \vec{v}(Q_i)}{\partial \theta_2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \vec{v}(Q_i)}{\partial \alpha} & \frac{\partial \vec{v}(Q_i)}{\partial \beta} & \frac{\partial \vec{v}(Q_i)}{\partial \theta_2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \theta_2} \vec{v}(Q_i)
\end{bmatrix}
\]

By the use of 10.5.5(4) in Lütkepohl (1996), the second summand of (1.4) equals
\[
\left( D_{p,-}^+ \otimes \frac{\partial \vec{v}(Q_i)'}{\partial \theta_2} \frac{\partial \vec{v}(Q_i)'}{\partial \theta_2'} D_p \right)^t \frac{\partial \vec{v}(Q_i) \otimes Q_i}{\partial \theta_2'} + \left( D_{p,-}^+ (Q_i \otimes Q_i) D_p \otimes I_{p^2} \right) \frac{\partial}{\partial \theta_2} \vec{v} \left( \frac{\partial \vec{v}(Q_i)'}{\partial \theta_2} \right)
\]
with
\[
\frac{\partial \vec{v}(Q_i) \otimes Q_i}{\partial \theta_2'} = \left( I_p \otimes K_{pp} \otimes I_p \right) \left[ \frac{\partial \vec{v}(Q_i)'}{\partial \theta_2'} \otimes \vec{v}(Q_i) + \vec{v}(Q_i) \otimes \frac{\partial \vec{v}(Q_i)'}{\partial \theta_2'} \right]
\]
and
\[
\frac{\partial \vec{v}(Q_i)'}{\partial \theta_2} = \left( I_p \otimes Q_i \right) \frac{\partial \vec{v}(Q_i)'}{\partial \theta_2'} + \left( Q_i \otimes I_p \right) \frac{\partial \vec{v}(Q_i)'}{\partial \theta_2'}
\]

\[
= \left( I_p \otimes Q_i \right) D_p \frac{\partial \vec{v}(Q_i)'}{\partial \theta_2'} + \left( Q_i \otimes I_p \right) D_p \frac{\partial \vec{v}(Q_i)'}{\partial \theta_2'}.
\]
2. The Proofs of the Theoretical Statements

2.1 The Proof of Proposition 4.1

The proof of Proposition 4.1 is organized as follows. The statement \( \hat{D}_m \overset{a.s.}{\to} D \) corresponds to

\[
\left| \hat{D}_m \left( \hat{\theta}_m \right) - D (\theta) \right| \overset{a.s.}{\to} 0
\]

(2.1)

For the lefthand side of (2.1), we have

\[
\left| \hat{D}_m \left( \hat{\theta}_m \right) - D (\theta) \right| \leq \left| \hat{D}_m \left( \hat{\theta}_m \right) - D_m \left( \hat{\theta}_m \right) \right| + \left| D_m \left( \hat{\theta}_m \right) - D_m (\theta) \right| + \left| D_m (\theta) - D (\theta) \right|
\]

(2.2)

The proof is based on the fact that all summands on the righthand side of (2.2) converge to zero almost surely. For the first summand that is to prove that the variation matrix which is calculated from a finite past of observations is a suitable substitute for the matrix based on an infinite past of observations. Hence, it has to be shown that we have for \( m \to \infty \)

\[
\sup_{u \in U} \left| \hat{D}_m (u) - D_m (u) \right| \overset{a.s.}{\to} 0
\]

(2.3)

with

\[
\hat{D}_m (u) = \frac{1}{m} \sum_{t=1}^{m} \hat{l}_t (u) \hat{l}_t (u)^T
\]

Since \( \hat{\theta}_m \) is a consistent estimator under Assumptions 3.7 and 3.8, the second summand in (2.2) converges to zero with the continuous mapping theorem. The same applies to the third summand, if the limit matrix

\[
D (u) = \mathbb{E} \left[ l_0' (u) l_0' (u)^T \right]
\]

exists for all \( u \in U \) and \( D_m (\cdot) \) converges uniformly to \( D (\cdot) \) in \( U \), that is

\[
\sup_{u \in U} \left| D_m (u) - D (u) \right|.
\]

Hence, we have to prove

\[
\mathbb{E} \left[ \sup_{u \in U} \left| l_0' (u) l_0' (u)^T \right| \right] < \infty.
\]

(2.4)

Then, the proposition follows from the almost sure convergence of the righthand side of (2.2) to zero or the validity of (2.3) and (2.4), respectively.
2.1.1 The Proof of $\sup_{u \in U} |\hat{D}_m(u) - D_m(u)| \xrightarrow{d, a.s.} 0$

Along the lines of Berkes et al. (2003), we use a multivariate version of the classic mean value theorem throughout the proof section:

**Lemma 2.1.** Let $f : \mathbb{R}^l \rightarrow \mathbb{R}$ be a function that is continuous in all of its arguments. Furthermore, let $x$ and $y$ be some $l$ dimensional real valued vectors. Then, there exists a vector $\xi \in \mathbb{R}^l$ with $|\xi - x| \leq |x - y|$ and $|\xi - y| \leq |x - y|$, such that

$$|f(x) - f(y)| = \left| \frac{\partial f(x)}{\partial x} \right|_{x=\xi} |x - y|. \quad (2.5)$$

**Proof:** Denote $x := (x_1, \ldots, x_l)'$, $y := (y_1, \ldots, y_l)'$ and $y_i := (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_l)'$. Note that $|x - y| = |x_i - y_i|$. A componentwise application of the univariate mean value theorem implies that for all $i \in \{1, \ldots, l\}$ there exist $\xi_i \in [x_i, y_i]$ and $\tilde{\xi}_i := (x_1, \ldots, x_{i-1}, \xi_i, x_{i+1}, \ldots, x_l)'$ with

$$|f(x) - f(y_i)| = \left| \frac{\partial f(x)}{\partial x_i} \right|_{x=\xi_i} |x_i - y_i|.$$ 

Choosing $\tilde{\xi} := (\tilde{\xi}_1, \ldots, \tilde{\xi}_l)'$ yields

$$|\tilde{\xi}_i - x| \leq |\tilde{\xi}_i - y_i| \leq |x_i - y_i| \quad \text{and} \quad |\tilde{\xi}_i - y| \leq |x_i - y_i| \quad \forall \ i \in \{1, \ldots, p\}$$

and also (2.5), which completes the proof.

For the lefthand side of (2.3) Lemma 2.1 yields

$$\sup_{u \in U} |\hat{D}_m(u) - D_m(u)| = \frac{1}{m} \sup_{u \in U} \left| \sum_{i=1}^{m} \left( \hat{P}_i(u) - P_i(u) \right) \right| \leq 2d \sup_{u \in U} \sup_{u \in \mathbb{Z}} |P_i(u)| \sup_{u \in U} \left| \sum_{i=1}^{m} \left( \hat{P}_i(u) - P_i(u) \right) \right| \quad (2.6)$$

where $v_i(u) \in \mathbb{R}^d$ is such that $|v_i(u) - P_i(u)| \leq |\hat{P}_i(u) - P_i(u)|$ and $|v_i(u) - \hat{P}_i(u)| \leq |\hat{P}_i(u) - P_i(u)|$.

Furthermore, for the sum on the righthand side of (2.6), we have

$$\sup_{u \in U} \left| \sum_{i=1}^{m} \left( \hat{P}_i(u) - P_i(u) \right) \right| = \max \left\{ \max_{1 \leq j \leq p} \left| \sum_{i=1}^{m} \left( \frac{\partial P_i(u)}{\partial u_j} - \frac{\partial \hat{P}_i(u)}{\partial u_j} \right) \right|, \sup_{u \in U} \left| \sum_{i=1}^{m} \left( \frac{\partial \hat{P}_i(u)}{\partial u_2} - \frac{\partial \hat{P}_i(u)}{\partial u_2} \right) \right| \right\}$$

where $u_1 := (r_1', \ldots, r_p')$ with $r_j := (x_j, s_j, t_j)'$, $j = 1, \ldots, p$ is composed of the variance parameters and $u_2 := (a, b, q_1, \ldots, q_p)'$ of the correlation parameters.
The Proof of \( \sup_{u \in U} \left| \sum_{t=1}^{m} \left( \frac{\partial l_t(u)}{\partial u} - \frac{\partial l_t(u)}{\partial u_1} \right) \right|^{a.s.} = O(1) \).

Throughout the next sections, we will use the following statement repeatedly. The lemma is a generalisation of Lemma 2.2 in Berkes et al. (2003). Adopting their notation, we define

\[
\log^+ x := \begin{cases} 
\log x, & x > 1 \\
0, & \text{else} 
\end{cases}
\]

**Lemma 2.2.** Let \( \{X_t, \ t \in \mathbb{N}_0\} \) be a sequence of identically distributed but not necessarily independent random variables satisfying

\[
\mathbb{E} \left[ \log^+ |X_0| \right] < \infty. \tag{2.7}
\]

Then, \( \sum_{k=0}^{\infty} k^j a^k X_k \) converges with probability one for any \( a \in \mathbb{R} \) with \( |a| < 1 \) and any fixed \( j \in \mathbb{N}_0 \).

**Proof:** Analogously to the proof of Lemma 2.2 in Berkes et al. (2003), it suffices to show that the conditions of the Borel-Cantelli Lemma are satisfied for all \( \zeta > 1 \). Note that the function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( f(x) = \frac{\zeta^x}{x^j} \) has a minimum located at \( x_{\min} = \frac{1}{\log \zeta} \) and is strictly monotonic increasing for larger values of \( x \). Thus, there exists some constant \( \zeta_0 \in (1, \zeta) \) with \( \frac{\zeta^k}{k^j} > \frac{\zeta_0^k}{k^j} \) for any integer \( k \geq k_0 \), where \( k_0 \) is the smallest integer that is larger than \( x_{\min} \). Thus, for our counterpart of (2.5) in Berkes et al. (2003), we have

\[
\sum_{k=0}^{\infty} P\left( |X_k| > \frac{\zeta^k}{k^j} \right) \leq \sum_{k=0}^{k_0-1} P\left( |X_k| > \frac{\zeta^k}{k^j} \right) + \sum_{k=k_0}^{\infty} P\left( |X_k| > \frac{\zeta^k}{k^j} \right) \leq k_0 + \sum_{k=0}^{\infty} P\left( |X_k| > \frac{\zeta_0^k}{k^j} \right). \tag{2.8}
\]

Along the lines of Berkes et al. (2003), the righthand side of (2.8) is finite if (2.7) is satisfied. Hence, the Borel-Cantelli Lemma implies the almost sure convergence for any nonnegative integer \( j \), which completes the proof.

**Lemma 2.3.** Denote \( T_{jt} := \sum_{k=t-1}^{\infty} x_j^2 y_{j,t-k-1} \) for \( j = 1, \ldots, p \) and \( t = 1, \ldots, m \). Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have for \( m \to \infty \):

\[
\sum_{t=1}^{m} T_{jt} \overset{a.s.}{=} O(1) \quad \text{and} \quad \sum_{t=1}^{m} T_{it} y_{it} y_{jt} \overset{a.s.}{=} O(1) \ \forall \ i, j \in \{1, \ldots, p\}.
\]

**Proof:** Assumptions 2.1, 3.1, 3.2 and 3.3 imply that both \( \{y_{jt}^2, \ t \in \mathbb{Z}\} \) and \( \{y_{it} y_{jt}, \ t \in \mathbb{Z}\} \) are sequences of unconditionally identically distributed random variables.
With Assumptions 3.5 and 3.6, we have for all \( i, j \in \{1, \ldots, p\} \):

\[
E \left[ \log^+ \left( y_{ij}^2 \right) \right] \leq E \left[ y_{ij}^2 \right] < \infty \quad \text{and} \quad E \left[ \log^+ \left( y_{ij} y_{ji} \right) \right] \leq E \left[ \log^+ \left( y_{ij}^2 y_{ji}^2 \right) \right] \leq E \left[ y_{ij}^2 y_{ji}^2 \right] < \infty.
\] (2.9)

Hence, Lemma 2.2 yields \( T_{j} \overset{a.s.}{=} O(1), \forall \ j \in \{1, \ldots, p\} \) and \( t \in \mathbb{Z} \). Additionally, for any \( i, j \in \{1, \ldots, p\} \) and for \( m \to \infty \), we have

\[
\sum_{t=1}^{m} T_{j} = \sum_{t=1}^{m} \sum_{l=-t}^{t} \rho^l y_{j+l-1}^2 = \sum_{t=1}^{m} \sum_{l=0}^{\infty} \rho^l y_{j}^2 = \frac{1 - \rho^m}{1 - \rho} \sum_{l=0}^{\infty} \rho^l y_{j}^2 \overset{a.s.}{=} O(1)
\]

and

\[
\left| \sum_{t=1}^{m} T_{i} y_{i} y_{j} \right| = \left| \sum_{t=1}^{m} y_{i} y_{j} \sum_{l=0}^{\infty} \rho^{l+1} y_{j+l}^2 \right| \leq \left| \sum_{t=0}^{m-1} \rho^l y_{i} y_{j+l+1} \right| \cdot \left| \sum_{l=0}^{\infty} \rho^l y_{j+l}^2 \right| \overset{a.s.}{=} O(1),
\]

where the last equality is implied by Lemma 2.2 in both cases.

\[\blacksquare\]

**Lemma 2.4.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have for all \( j \in \{1, \ldots, p\} \) and for \( m \to \infty \):

- \( \sup_{u \in \mathcal{U}} \left| \sum_{l=1}^{m} \left( \tilde{w}_{j}(u)^{-\frac{1}{2}} - w_{j}(u)^{-\frac{1}{2}} \right) \right| \overset{a.s.}{=} O(1) \);
- \( \sup_{u \in \mathcal{U}} \left| \sum_{l=1}^{m} \left( \tilde{w}_{j}(u)^{-\frac{1}{2}} - w_{j}(u)^{-\frac{1}{2}} \right) \right| \overset{a.s.}{=} O(1) \); and
- \( \sup_{u \in \mathcal{U}} \left| \sum_{l=1}^{m} \left( \tilde{w}_{j}(u) \tilde{w}_{j}(u)^{-\frac{1}{2}} - \left( w_{j}(u) w_{j}(u) \right)^{-\frac{1}{2}} \right) \right| \overset{a.s.}{=} O(1) \).

**Proof:** Analogously to the arguments in the proof of Lemma 5.4 in Berkes et al. (2003), there exist positive constants \( C_1 := \frac{u}{1 - \rho} \) and \( C_2 := \frac{u}{1 - \rho} \) with

\[
0 < C_1 \leq \tilde{w}_{j}(u) \leq w_{j}(u) \leq C_2 \left( 1 + \sum_{k=0}^{\infty} \rho^k y_{j+k-1}^2 \right) \overset{a.s.}{=} O(1) \quad \forall \ j \in \{1, \ldots, p\}.
\] (2.10)

With \( \tilde{w}_{j}(u) = \min \left[ \tilde{w}_{j}(u), w_{j}(u) \right] \), the mean value theorem yields for any \( j \in \{1, \ldots, p\} \)

\[
\sup_{u \in \mathcal{U}} \left| \sum_{l=1}^{m} \left( \tilde{w}_{j}(u)^{-\frac{1}{2}} - w_{j}(u)^{-\frac{1}{2}} \right) \right| \leq \sup_{u \in \mathcal{U}} \sum_{l=1}^{m} \frac{1}{2 \tilde{w}_{j}(u)} \left( w_{j}(u) - \tilde{w}_{j}(u) \right) \leq \frac{1}{2C_1} \sum_{l=1}^{m} T_{j} \overset{a.s.}{=} O(1).
\] (2.11)

For the second statement, (2.10) and (2.11) yield

\[
\sup_{u \in \mathcal{U}} \left| \sum_{l=1}^{m} \left( \tilde{w}_{j}(u)^{-\frac{1}{2}} - w_{j}(u)^{-\frac{1}{2}} \right) \right| \leq \sup_{u \in \mathcal{U}} \sum_{l=1}^{m} \frac{1}{\tilde{w}_{j}(u)} \left( w_{j}(u)^{\frac{1}{2}} \tilde{w}_{j}(u)^{-\frac{1}{2}} \right) \overset{a.s.}{=} O(1).
\]
Finally, with the mean value theorem and (2.10), we have for any $i, j \in \{1, \ldots, p\}$
\[
\sup_{u \in U} \sum_{t=1}^{m} \left[ (\tilde{w}_{ij}(u) \tilde{w}_{ij}(u)) - (w_{ij}(u) w_{ij}(u)) \right]^{\frac{1}{2}} \leq \frac{1}{C_{1}^2} \sup_{u \in U} \sum_{t=1}^{m} \left[ (w_{ij}(u) - \tilde{w}_{ij}(u)) \tilde{w}_{ij}(u) + (w_{ij}(u) - \tilde{w}_{ij}(u)) w_{ij}(u) \right] \leq \frac{\rho}{2C_{1}^2} \sup_{u \in U} \sum_{t=1}^{m} (T_{ij} - T_{ij}) \leq \frac{\rho}{2C_{1}^2} \max_{1 \leq j \leq p} \sup_{u, t, i \in U} \sum_{t=1}^{m} T_{ij} \overset{a.s.}{=} O(1). \]

**Lemma 2.5.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have for any $j \in \{1, \ldots, p\}$:

- \( \sup_{u \in U, t, i \in \mathbb{Z}} \left| \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial t_{i}} \right| \overset{a.s.}{=} O(1) \); and

- \( \sup_{u \in U} \left| \sum_{t=1}^{m} \left( \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial t_{i}} - \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial t_{j}} \right) \right| \overset{a.s.}{=} O(1) \).

**Proof:** By the use of (2.9), (2.10) and Lemma 2.2, we have for any $j \in \{1, \ldots, p\}$ and $m \to \infty$:

- \( \sup_{u \in U, t, i \in \mathbb{Z}} \left| \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial t_{i}} \right| \overset{a.s.}{=} O(1) \); \quad \text{(2.12)}

- \( \sup_{u \in U, t, i \in \mathbb{Z}} \left| \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial s_{j}} \right| \overset{a.s.}{=} O(1) \); \quad \text{(2.13)}

- \( \sup_{u \in U, t, i \in \mathbb{Z}} \left| \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial t_{j}} \right| \overset{a.s.}{=} O(1) \); \quad \text{(2.14)}

Thus, the first statement is an immediate consequence of (2.12)-(2.14).

The second statement follows from (2.15)-(2.17) below. With Lemma 2.4, we have

\[
\sup_{u \in U} \left| \sum_{t=1}^{m} \left( \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial x_{j}} - \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial x_{j}} \right) \right| \overset{a.s.}{=} \frac{1}{2} \frac{1}{1 - \rho} \sup_{u \in U} \left| \sum_{t=1}^{m} (\tilde{w}_{ij}(u) - w_{ij}(u)) \right| \overset{a.s.}{=} O(1) \quad \text{(2.15)}
\]

Furthermore, (2.10) and Lemmas 2.3 and 2.4 yield

\[
\sup_{u \in U} \left| \sum_{t=1}^{m} \left( \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial s_{j}} - \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial s_{j}} \right) \right| \overset{a.s.}{=} O(1) \quad \text{(2.16)}
\]
Finally, we have
\[
\sup_{u \in U} \left| \sum_{t=1}^{m} \left( \frac{\partial \text{vec}(\tilde{F}_D(u))'}{\partial t_j} - \frac{\partial \text{vec}(F_D(u))'}{\partial t_j} \right) \right| \leq \frac{1}{2} C_1^{-1/2} \sum_{t=1}^{m} T_{jt}^{1/2} + \frac{1}{2} C_1^{-1/2} \sum_{t=1}^{m} T_{jt}^{1/2} \overset{a.s.}{=} \mathcal{O}(1). \quad (2.16)
\]

As (2.18) is $O(1)$ almost surely with (2.9) and Lemma 2.2, this also applies to (2.17), which completes the proof.

**Lemma 2.6.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have for $m \to \infty$:

- $\sup_{u \in U} \left| \sum_{t=1}^{m} (\tilde{F}_D(u) - F_D(u)) \right| \overset{a.s.}{=} \mathcal{O}(1)$; and

- $\sup_{u \in U} \left| \sum_{t=1}^{m} (\tilde{F}_Q(u) - F_Q(u)) \right| \overset{a.s.}{=} \mathcal{O}(1)$.

**Proof:** For the first statement, Lemma 2.4 implies
\[
\sup_{u \in U} \left| \sum_{t=1}^{m} (\tilde{F}_D(u) - F_D(u)) \right| = \max_{1 \leq j \leq \rho} \left| \sum_{t=1}^{m} (\tilde{w}_j(u))^{1/2} - w_j(u)^{1/2} \right| \overset{a.s.}{=} \mathcal{O}(1).
\]

For the second statement, (2.9), (2.10) and Lemmas 2.2, 2.3 and 2.4 yield
\[
\sup_{u \in U} \left| \sum_{t=1}^{m} (\tilde{F}_Q(u) - F_Q(u)) \right| = \sup_{u \in U} \left| a \sum_{t=1}^{m} \sum_{k=1}^{\infty} \rho^{k-1} (\tilde{z}_t(u)\tilde{z}'_t(u) - z_t(u)z'_t(u)) \right|
\]
\[
\leq \sup_{u \in U} \max_{1 \leq i, j \leq \rho} \left| \sum_{k=1}^{\infty} \rho^{k-1} y_{i,t-k} y_{j,t-k} \left[ \left( \tilde{w}_{i,t-k}(u) \tilde{w}_{j,t-k}(u) \right)^{1/2} - \left( w_{i,t-k}(u) w_{j,t-k}(u) \right)^{1/2} \right] \right|
\]
\[
\leq \sup_{u \in U} \max_{1 \leq i, j \leq \rho} \left| \sum_{k=1}^{\infty} \rho^{k-1} y_{i,t-k} y_{j,t-k} \left[ \left( \tilde{w}_{i,t-k}(u) w_{j,t-k}(u) \right)^{1/2} - \left( w_{i,t-k}(u) w_{j,t-k}(u) \right)^{1/2} \right] \right| \overset{a.s.}{=} \mathcal{O}(1).
\]

In the following, the proofs for terms with finite and infinite past work analogously and will be omitted for one of these cases.

**Lemma 2.7.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have:
Proof: Statement (2.10) and Lemma 2.2 of Berkes et al. (2003) yield
\[
\sup_{u \in U; t \in \mathbb{Z}} |F_D(u)|^{a.s.} = O(1), \quad \sup_{u \in U; t \in \mathbb{Z}} |F_Q(u)|^{a.s.} = O(1), \text{ and } \sup_{u \in U; t \in \mathbb{Z}} |F_R(u)|^{a.s.} = O(1); \text{ and}
\]
\[
\sup_{u \in U; t \in \mathbb{Z}} \left| \tilde{F}_D(u) \right|^{a.s.} = O(1), \quad \sup_{u \in U; t \in \mathbb{Z}} \left| \tilde{F}_Q(u) \right|^{a.s.} = O(1), \text{ and } \sup_{u \in U; t \in \mathbb{Z}} \left| \tilde{F}_R(u) \right|^{a.s.} = O(1).
\]

Concludingly, \(F_R(u)\) is a correlation matrix for all \(t \in \mathbb{Z}\) almost surely. Hence, the absolute entries are bounded by 1 almost surely, which completes the proof.

In the following, denote by \(S_n\) the set of all \(n!\) permutations of \((1, \ldots, n)\) and \(\text{sgn}(\pi)\) the sign of the permutation \(\pi = (\pi(1), \ldots, \pi(n))\) that indicates, whether an even or odd number of pairwise interchanges of neighbouring entries in \((1, \ldots, n)\) is necessary to obtain the permutation \(\pi\).

**Lemma 2.8.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have:

- \(\sup_{u \in U; t \in \mathbb{Z}} \left| \det F_Q(u) \right|^{a.s.} = O(1)\);
- \(\sup_{u \in U; t \in \mathbb{Z}} \left| \det \tilde{F}_Q(u) \right|^{a.s.} = O(1)\); and
- \(\sup_{u \in U; t \in \mathbb{Z}} \left\{ \sum_{r=1}^m \left( \det \tilde{F}_Q(u) - \det F_Q(u) \right) \right\}^{a.s.} = O(1)\).

**Proof:** Note that the determinant of a matrix equals the products of its eigenvalues. Hence, Assumption 2.1(6) implies
\[
\sup_{u \in U; t \in \mathbb{Z}} \left| \det F_Q(u) \right| \leq \left( \sup_{u \in U; t \in \mathbb{Z}} \lambda_{\text{max}}(F_Q(u)) \right)^{p} = \delta_2^p.
\]
For the second statement, Lemmas 2.6 and 2.7 yield
\[
\sup_{u \in U} \left| \sum_{i=1}^m \left( \det \tilde{F}_{Q_i}(u) - \det F_{Q_i}(u) \right) \right|
\]
\[
= \sup_{u \in U} \left| \sum_{i=1}^m \sum_{p \in S_p} \left[ \left( \det F_{Q_i}(u)_{1 \pi(1)} \right) \cdots \left( \det F_{Q_i}(u)_{p \pi(p)} \right) - \left( \det F_{Q_i}(u)_{1 \pi(1)} \right) \cdots \left( \det F_{Q_i}(u)_{p \pi(p)} \right) \right] \right|
\]
\[
= \sup_{u \in U} \left| \sum_{i=1}^m \sum_{p \in S_p} \sum_{j=1}^{p-1} \left[ \left( \det F_{Q_i}(u)_{j \pi(j)} \right) - \left( \det F_{Q_i}(u)_{j \pi(j)} \right) \right] \prod_{i=1}^{j-1} \left[ \det F_{Q_i}(u)_{i \pi(i)} \right] \prod_{k=j+1}^{p} \left[ \det F_{Q_i}(u)_{k \pi(k)} \right] \right|
\]
\[
\leq p! (p-1) \sup_{u \in U} \left| \sum_{i=1}^m \left( F_{Q_i}(u) - \tilde{F}_{Q_i}(u) \right) \right| \max_{1 \leq j \leq p} \left[ \sup_{u \in U, i \in Z} \left| F_{Q_i}(u) \right| \right] \left( \sup_{u \in U, i \in Z} \left| \det F_{Q_i}(u) \right| \right)^{p-j} \overset{a.s.}{=} O(1).
\]

**Lemma 2.9.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have:

- \( \sup_{u \in U, i \in Z} \det F_{Q_i}(u) \overset{a.s.}{=} O(1) \);

- \( \sup_{u \in U, i \in Z} \det \tilde{F}_{Q_i}(u) \overset{a.s.}{=} O(1) \); and

- \( \sup_{u \in U} \left| \sum_{i=1}^m \left( \det F_{Q_i}(u) \right)^2 - \left( \det \tilde{F}_{Q_i}(u) \right)^2 \right| \overset{a.s.}{=} O(1) \).

**Proof:** For the first statement, we have
\[
\sup_{u \in U, i \in Z} \det \tilde{F}_{Q_i}(u) = \sup_{u \in U, i \in Z} \prod_{i=1}^p \left| \tilde{F}_{Q_i}(u)_{i \pi(i)} \right|^{1/2} \leq \sup_{u \in U} \left( \frac{1-a-b}{1-b} \right)^{p/2} \leq \left( \frac{1-a}{1-b} \right)^{p/2} = O(1).
\]

For the second statement, consider the interval \( I_{Q_i}(u) := \left[ \xi^*_i(u), \tilde{\xi}^*_i(u) \right] \) with
\[
\xi^*_i(u) := \min \left\{ \det \tilde{F}_{Q_i}(u), \det F_{Q_i}(u) \right\} \quad \text{and} \quad \tilde{\xi}^*_i(u) := \max \left\{ \det \tilde{F}_{Q_i}(u), \det F_{Q_i}(u) \right\}.
\]

Hence, with the mean value theorem and for any \( t \in \mathbb{Z} \) and \( u \in U \), there exists a \( \tilde{\xi}^*_i(u) \in I_{Q_i}(u) \) such that Lemmas 2.6, 2.9 and (2.8) from the paper yield
\[
\sup_{u \in U} \left| \sum_{i=1}^m \left( \det \tilde{F}_{Q_i}(u) - \det F_{Q_i}(u) \right)^2 \right| = \sup_{u \in U} \sum_{i=1}^m 2 \tilde{\xi}^*_i(u) \left| \det \tilde{F}_{Q_i}(u) - \det F_{Q_i}(u) \right|
\]
\[
\leq 2 \sup_{u \in U, i \in Z} \sup_{u \in U} \left| \sum_{i=1}^m \left[ \prod_{i=1}^p \left| F_{Q_i}(u)_{i \pi(i)} \right|^{1/2} - \prod_{i=1}^p \left| F_{Q_i}(u)_{i \pi(i)} \right|^{1/2} \right] \right|
\]
\[
= 2 \sup_{u \in U, i \in Z} \sup_{u \in U} \left| \sum_{i=1}^m \sum_{j=1}^p \left( F_{Q_i}(u)_{j \pi(j)}^{1/2} - F_{Q_i}(u)_{j \pi(j)}^{1/2} \right) \prod_{i=1}^{j-1} \left| F_{Q_i}(u)_{i \pi(i)} \right| \prod_{k=j+1}^p \left| F_{Q_i}(u)_{k \pi(k)} \right| \right|
\]
\[
\overset{\text{MVT}}{\leq} 2 \sup_{u \in U, i \in Z} \sup_{u \in U} \sum_{j=1}^p \sum_{i=1}^m \frac{1}{2} \left( F_{Q_i}(u)_{j \pi(j)}^{1/2} - F_{Q_i}(u)_{j \pi(j)}^{1/2} \right) \sup_{u \in U} \left( \frac{1-a-b}{1-b} \right)^{p-j} \overset{a.s.}{=} O(1).
\]
Lemma 2.10. Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have:

\begin{itemize}
  \item $\sup_{u \in U, \epsilon \in \mathbb{Z}} \frac{1}{\det F_R(u)} \overset{a.s.}{=} O(1)$,
  \item $\sup_{u \in U, \epsilon \in \mathbb{Z}} \frac{1}{\det F_R(u)} \overset{a.s.}{=} O(1)$; and
  \item $\sup_{u \in U, \epsilon \in \mathbb{Z}} \left| \sum_{i=1}^{m} \left( \frac{1}{\det F_R(u)} - \frac{1}{\det F_R(u)} \right) \right| \overset{a.s.}{=} O(1)$.
\end{itemize}

Proof: For the inverted determinants, statement (2.9) from the paper yields

$$
\sup_{u \in U, \epsilon \in \mathbb{Z}} \det F_R(u) \geq \left( \sup_{u \in U, \epsilon \in \mathbb{Z}} \lambda_{\min}(F_R(u)) \right)^p \overset{a.s.}{>} \left[ \frac{1 - a - b \delta_1}{p\delta_2} \right]^p \Rightarrow \sup_{u \in U, \epsilon \in \mathbb{Z}} \det F_R(u) \overset{a.s.}{\leq} \left[ \frac{1 - b \delta_2}{1 - a - b \delta_1} \right]^p = O(1).
$$

For the last statement of the lemma, consider $I_R(u) := \left[ \xi(u), \bar{\xi}(u) \right]$ with

$$
\xi(u) := \min \left\{ \det \bar{F}_R(u), \det F_R(u) \right\} \quad \text{and} \quad \bar{\xi}(u) := \max \left\{ \det \bar{F}_R(u), \det F_R(u) \right\}.
$$

Thus, with the mean value theorem and for any $t \in \mathbb{Z}$ and $u \in U$, there exists a $\xi_t(u) \in I_R(u)$ such that Lemmas 2.8 and 2.9 imply

\begin{align*}
\sup_{u \in U} \left| \sum_{i=1}^{m} \left( \frac{1}{\det \bar{F}_R(u)} - \frac{1}{\det F_R(u)} \right) \right| &\leq \sup_{u \in U, \epsilon \in \mathbb{Z}} \left| - \frac{1}{\xi_t(u)} \sup_{u \in U} \sum_{i=1}^{m} \left( \det \bar{F}_R(u) - \det F_R(u) \right) \right| \\
&\leq \sup_{u \in U, \epsilon \in \mathbb{Z}} \frac{1}{\xi_t(u)} \sup_{u \in U} \sum_{i=1}^{m} \left( \left( \det \bar{F}_Q(u) \right)^2 - \left( \det F_Q(u) \right)^2 \right) \\
&\leq \sup_{u \in U, \epsilon \in \mathbb{Z}} \frac{1}{\xi_t(u)} \sup_{u \in U} \left( \det \bar{F}_Q(u) \right) \sum_{i=1}^{m} \left( \left( \det \bar{F}_Q(u) \right)^2 - \left( \det F_Q(u) \right)^2 \right) \\
&\quad + \sup_{u \in U, \epsilon \in \mathbb{Z}} \frac{1}{\xi_t(u)} \left( \sup_{u \in U} \sum_{i=1}^{m} \left( \det \bar{F}_Q(u) - \det F_Q(u) \right) \right) \overset{a.s.}{=} O(1).
\end{align*}

Let $X^{(i,j)}$ be the matrix that results from $X \sim (n \times n)$ by omitting the $i$-th row and the $j$-th column with $1 \leq i, j \leq n$.

Lemma 2.11. Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have:

\begin{itemize}
  \item $\sup_{u \in U, \epsilon \in \mathbb{Z}} \det F_R(u)^{(i,j)} \overset{a.s.}{=} O(1)$,
  \item $\sup_{u \in U, \epsilon \in \mathbb{Z}} \det \bar{F}_R(u)^{(i,j)} \overset{a.s.}{=} O(1)$; and
  \item $\sup_{u \in U} \left| \sum_{i=1}^{m} \left( \det \bar{F}_R(u)^{(i,j)} - \det F_R(u)^{(i,j)} \right) \right| \overset{a.s.}{=} O(1)$.
\end{itemize}
Proof: The proof works analogously to the proof of Lemma 2.8 and uses the arguments in the proof of Lemma 2.10.

**Lemma 2.12.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have:

\[
\sup_{u \in U} \left| \sum_{i=1}^{m} (\tilde{F}_{D_i}(u)^{-1} - F_{D_i}(u)^{-1}) \right|_\infty \leq O(1) \quad \text{and} \quad \sup_{u \in U} \left| \sum_{i=1}^{m} (\tilde{F}_{R_i}(u)^{-1} - F_{R_i}(u)^{-1}) \right|_\infty \leq O(1).
\]

Proof: The Lemmas 2.4, 2.10 and 2.11 imply

\[
\sup_{u \in U} \left| \sum_{i=1}^{m} (\tilde{F}_{D_i}(u)^{-1} - F_{D_i}(u)^{-1}) \right| = \sup_{u \in U} \max_{1 \leq i \leq p} \left| \sum_{i=1}^{m} \left( \tilde{w}_{ij}(u)^{-1/2} - w_{ij}(u)^{-1/2} \right) \right|^{a_s} \leq O(1)
\]

and

\[
\sup_{u \in U} \left| \sum_{i=1}^{m} (\tilde{F}_{R_i}(u)^{-1} - F_{R_i}(u)^{-1}) \right| = \sup_{u \in U} \max_{1 \leq i \leq p} \left| \sum_{i=1}^{m} \left( \tilde{w}_{ij}(u)^{-1/2} - w_{ij}(u)^{-1/2} \right) \right|^{a_s} \leq O(1).
\]

**Lemma 2.13.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have:

- \(\sup_{u \in U, r \in Z} |F_{D_i}(u)^{1/2}| \leq O(1), \quad \sup_{u \in U, r \in Z} |F_{Q_i}(u)^{1/2}| \leq O(1), \quad \sup_{u \in U, r \in Z} |F_{O_i}(u)^{1/2}| \leq O(1);\) and

- \(\sup_{u \in U, r \in Z} |\tilde{F}_{D_i}(u)^{1/2}| \leq O(1), \quad \sup_{u \in U, r \in Z} |\tilde{F}_{Q_i}(u)^{1/2}| \leq O(1), \quad \sup_{u \in U, r \in Z} |\tilde{F}_{O_i}(u)^{1/2}| \leq O(1).\)

Proof: First, (2.10) implies \(\sup_{u \in U, r \in Z} |F_{D_i}(u)^{1/2}| = \sup_{u \in U, r \in Z} |\tilde{F}_{D_i}(u)^{1/2}| \leq O(1).\)

Next, consider the eigenvalue decomposition of \(F_{Q_i}(u)^{1/2}\) with \(U_i(u)\) the matrix whose columns are the orthonormalized eigenvectors that belong to the ordered eigenvalues of \(F_{Q_i}(u)\) which form the main diagonal of the diagonal matrix \(\Lambda_i(u)\). Note that due to the normalization we have \(\sup_{u \in U, r \in Z} |U_i(u)| \leq 1\) and that

\[
\sup_{u \in U, r \in Z} |F_{Q_i}(u)^{1/2}| \leq \sup_{u \in U} \left| 1 - a - b \right| \min_{1 \leq i \leq p} \left| F_{Q_i}(u) \right| \leq \left( \frac{1 - a}{1 - 2b} \right)^{1/4} = O(1).
\]

and

\[
\sup_{u \in U, r \in Z} |F_{O_i}(u)^{1/2}| \leq \sup_{u \in U} \left| 1 - a - b \right| \min_{1 \leq i \leq p} \left| F_{Q_i}(u) \right| \leq \left( \frac{1 - a}{1 - 2b} \right)^{1/4} = O(1).
\]
Lemma 2.14. Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have

- \( \sup_{u \in U, i \in \mathbb{Z}} |F_D(u)^{-1}| \overset{a.s.}{=} O(1) \) and \( \sup_{u \in U, i \in \mathbb{Z}} |F_R(u)^{-1}| \overset{a.s.}{=} O(1) \); and

- \( \sup_{u \in U, i \in \mathbb{Z}} |\widetilde{F}_D(u)^{-1}| \overset{a.s.}{=} O(1) \) and \( \sup_{u \in U, i \in \mathbb{Z}} |\widetilde{F}_R(u)^{-1}| \overset{a.s.}{=} O(1) \).

Proof: Note that (2.10) implies

\[
\sup_{u \in U, i \in \mathbb{Z}} |F_D(u)^{-1}| = \sup_{u \in U, i \in \mathbb{Z}} \max_{1 \leq j \leq p} w_j(u)^{-1/2} \leq C_1^{-1/2} \overset{a.s.}{=} O(1).
\]

To prove \( \sup_{u \in U, i \in \mathbb{Z}} |F_R(u)^{-1}| \overset{a.s.}{=} O(1) \), we investigate the matrix \( F_R(u)^{-1} \) in detail. For this purpose, keep in mind that \( F_R(u)^{-1} = [\det F_R(u)]^{-1} A_i(u) \) with \( A_i(u) := (a_{ij}(u))_{i,j=1,...,p} \) the adjoint matrix and \( a_{ij}(u) \) the cofactor of \( [F_R(u)]_{ij} \) which is defined as \( a_{ij}(u) := (-1)^{i+j} M_{ij}(u) \) with \( M_{ij}(u) := \det F_R(u)^{(i,j)} \) the minor of \( [F_R(u)]_{ij} \). Note that since the entries of \( F_R(u) \) do not exceed one in modulus, \( M_{ij}(u) \) is bounded by a constant:

\[
\sup_{u \in U, i \in \mathbb{Z}} \max_{1 \leq j \leq p} |M_{ij}(u)| \leq \sup_{u \in U, i \in \mathbb{Z}} \max_{1 \leq j \leq p \leq r \leq p-1} \prod_{k=1}^{p-1} |[F_R(u)]^{(i,j)}(k,k)| < (p - 1)!.
\]

Thus, analogously to the argumentation in Lemma 2.10, (2.9) in the paper and (2.19) above yield

\[
\sup_{u \in U, i \in \mathbb{Z}} \frac{1}{\det F_R(u)} \max_{1 \leq j \leq p} \sup_{u \in U, i \in \mathbb{Z}} |\det F_R(u)^{(i,j)}| \overset{a.s.}{=} \left\lfloor \frac{1 - b}{1 - a - b} \right\rfloor^p (p - 1)!,
\]

which completes the proof.

Lemma 2.15. Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have

- \( \sup_{u \in U, i \in \mathbb{Z}} |Z_i(u)Z_i(u')| \overset{a.s.}{=} O(1) \),

- \( \sup_{u \in U, i \in \mathbb{Z}} |\overline{Z}_i(u)\overline{Z}_i(u')| \overset{a.s.}{=} O(1) \); and

- \( \sup_{u \in U} \left| \sum_{i=1}^m \left( Z_i(u)\overline{Z}_i(u') - Z_i(u)Z_i(u') \right) \right| \overset{a.s.}{=} O(1) \).

Proof: With \( \mathcal{F}_H(u) := \mathcal{F}_D(u)\mathcal{F}_R(u)\mathcal{F}_D(u) \) and by the use of the Lemmas 2.13 and 2.14 as well as Assumptions 3.5 and 3.6 which imply the existence of the moments of the innovations, we have

\[
\sup_{u \in U, i \in \mathbb{Z}} |Z_i(u)Z_i(u')| \leq p^2 \sup_{u \in U, i \in \mathbb{Z}} |F_D(u)^{-1}| \sup_{u \in U, i \in \mathbb{Z}} |F_R(u)^{-1}| \sup_{u \in U, i \in \mathbb{Z}} |\mathcal{F}_H(u)^{1/2} \epsilon \epsilon' \mathcal{F}_H(u)^{1/2}| \sup_{u \in U, i \in \mathbb{Z}} |F_D(u)^{-1}|
\]

\[
\leq p^4 \left( \sup_{u \in U, i \in \mathbb{Z}} |F_D(u)^{-1}| \right)^2 \left( \sup_{u \in U, i \in \mathbb{Z}} |F_D(u)|^{1/2} \right)^4 \left( \sup_{u \in U, i \in \mathbb{Z}} |F_R(u)|^{1/2} \right)^2 \sup_{i \in \mathbb{Z}} \epsilon \epsilon'
\]

\[
\leq p^6 \left( \sup_{u \in U, i \in \mathbb{Z}} |F_D(u)^{-1}| \right)^2 \left( \sup_{u \in U, i \in \mathbb{Z}} |F_D(u)|^{1/2} \right)^4 \left( \sup_{u \in U} |F_Q(u)|^{1/2} \right)^4 \left( \sup_{u \in U, i \in \mathbb{Z}} |F_Q(u)|^{1/2} \right)^2 \sup_{i \in \mathbb{Z}} \epsilon \epsilon' \overset{a.s.}{=} O(1).
\]
Furthermore, (2.10) and Lemma 2.3 yield

\[ \sup_{u \in U} \left| \sum_{t=1}^{m} \left( Z_t(u)Z_t(u)^* - z_t(u)z_t(u)^* \right) \right| = \sup_{u \in U} \max_{1 \leq i, j \leq p} \left| \sum_{t=1}^{m} y_{it}y_{jt} \left( \tilde{w}_{it}(u)\tilde{w}_{jt}(u) \right)^{-1} \right| \]

\[ \leq \frac{p}{2} \sup_{u \in U} \max_{1 \leq i, j \leq p} \left| \sum_{t=1}^{m} \left( T_{it}\tilde{w}_{it}(u) + T_{jt}\tilde{w}_{jt}(u) \right) y_{it}y_{jt} \right| \]

\[ \leq \frac{p}{2} \max_{1 \leq i, j \leq p} \sup_{u \in U} \sup_{t \in \mathbb{Z}} \left( \sum_{t=1}^{m} T_{it}y_{it}y_{jt} \right) + \left( \sum_{t=1}^{m} T_{jt}y_{it}y_{jt} \right) \overset{a.s.}{=} O(1). \]

Denote \( K_{1}(u) := \bar{F}_{D}(u)^{-1} + \bar{F}_{D}(u)^{-1}\bar{F}_{R}(u)^{-1}\bar{Z}_{t}(u)\bar{Z}_{t}(u)^* \) and analogously \( \hat{K}_{1}(u) \) in dependence of a finite past of observations.

**Lemma 2.16.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have

- \( \sup_{u \in U, t \in \mathbb{Z}} |K_{1}(u)| \overset{a.s.}{=} O(1) \),
- \( \sup_{u \in U, t \in \mathbb{Z}} |\hat{K}_{1}(u)| \overset{a.s.}{=} O(1) \); and
- \( \sup_{u \in U} \left| \sum_{t=1}^{m} \left( K_{1}(u) - \hat{K}_{1}(u) \right) \right| \overset{a.s.}{=} O(1) \).

**Proof:** The Lemmas 2.12, 2.14 and 2.15 imply

\[ \sup_{u \in U, t \in \mathbb{Z}} |K_{1}(u)| \leq \sup_{u \in U, t \in \mathbb{Z}} |F_{D}(u)^{-1}| + p^2 \sup_{u \in U, t \in \mathbb{Z}} |F_{D}(u)^{-1}| \sup_{u \in U, t \in \mathbb{Z}} |F_{R}(u)^{-1}| \sup_{u \in U, t \in \mathbb{Z}} |Z_{t}(u)Z_{t}(u)^*| \overset{a.s.}{=} O(1) \]

and

\[ \sup_{u \in U} \left| \sum_{t=1}^{m} \left( K_{1}(u) - \hat{K}_{1}(u) \right) \right| \leq p^2 \sup_{u \in U, t \in \mathbb{Z}} |\bar{F}_{R}(u)^{-1}| \sup_{u \in U, t \in \mathbb{Z}} |\bar{Z}_{t}(u)\bar{Z}_{t}(u)^*| \sup_{u \in U} \left| \sum_{t=1}^{m} \left( \bar{F}_{D}(u)^{-1} - F_{D}(u)^{-1} \right) \right| \]

\[ + p^2 \sup_{u \in U, t \in \mathbb{Z}} |F_{D}(u)^{-1}| \sup_{u \in U, t \in \mathbb{Z}} |\bar{Z}_{t}(u)\bar{Z}_{t}(u)^*| \sup_{u \in U} \left| \sum_{t=1}^{m} \left( \bar{F}_{R}(u)^{-1} - F_{R}(u)^{-1} \right) \right| \]

\[ + p^2 \sup_{u \in U, t \in \mathbb{Z}} |F_{D}(u)^{-1}| \sup_{u \in U, t \in \mathbb{Z}} |F_{R}(u)^{-1}| \sup_{u \in U} \left| \sum_{t=1}^{m} \left( \bar{Z}_{t}(u)\bar{Z}_{t}(u)^* - Z_{t}(u)Z_{t}(u)^* \right) \right| \overset{a.s.}{=} O(1) \]

**Lemma 2.17.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have for all \( j \in \{1, \ldots, p\} \)

\[ \sup_{u \in U, t \in \mathbb{Z}} \left| \frac{\partial \ell(u)}{\partial j} \right| \overset{a.s.}{=} O(1) \] and \( \sup_{u \in U} \left| \sum_{t=1}^{m} \left( \frac{\partial \ell(u)}{\partial j} - \frac{\partial \ell(u)}{\partial j} \right) \right| \overset{a.s.}{=} O(1) \).

**Proof:** The Lemmas 2.5, 2.12 and 2.16 imply

\[ \sup_{u \in U, t \in \mathbb{Z}} \left| \frac{\partial \ell(u)}{\partial j} \right| \leq \sup_{u \in U, t \in \mathbb{Z}} \left| \frac{\partial \text{vec}(F_{D}(u))^\prime}{\partial j} \right| \sup_{u \in U, t \in \mathbb{Z}} |K_{1}(u) \overset{a.s.}{=} O(1) \]
and \[
\sup_{u \in U} \left| \sum_{j=1}^{m} \left( \frac{\partial^2 f_i(u)}{\partial u_j} - \frac{\partial f_i(u)}{\partial u_j} \right) \right| = \sup_{u \in U} \left| \sum_{j=1}^{m} \left( \frac{\partial \text{vec}(F_{D_j}(u))}{\partial u_j} - \frac{\partial \text{vec}(F_{D_j}(u))}{\partial u_j} \right) \right| \leq p \sup_{u \in U, j \in \mathbb{Z}} |\hat{K}_{11}(u)| \sup_{u \in U} \left| \sum_{j=1}^{m} \left( \frac{\partial \text{vec}(F_{D_j}(u))}{\partial u_j} - \frac{\partial \text{vec}(F_{D_j}(u))}{\partial u_j} \right) \right| + p \sup_{u \in U, j \in \mathbb{Z}} \left| \text{vec}(F_{D_j}(u)) \right| \sup_{u \in U} \left| \sum_{j=1}^{m} (\hat{K}_{11}(u) - K_{11}(u)) \right| + p \sup_{u \in U, j \in \mathbb{Z}} \left| \text{vec}(F_{D_j}(u)) \right| \sup_{u \in U} \left| \sum_{j=1}^{m} (\hat{F}_{D_j}(u) - F_{D_j}(u)) \right| ^{a.s.} = O(1). \]

(II) The proof of \[
\sup_{u \in U} \left| \sum_{j=1}^{m} \left( \frac{\partial^2 f_i(u)}{\partial u_j^2} - \frac{\partial f_i(u)}{\partial u_j^2} \right) \right| ^{a.s.} = O(1). \]

Denote \(K_{21}(u) := F_{R}(u)^{-1} - F_{R}(u)^{-1} z_i(u) z_i(u)^T F_{R}(u)^{-1}\) and analogously \(\hat{K}_{21}(u)\) in dependence of a finite past of observations.

**Lemma 2.18.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have

- \(\sup_{u \in U, j \in \mathbb{Z}} |K_{21}(u)| ^ {a.s.} = O(1)\),
- \(\sup_{u \in U, j \in \mathbb{Z}} |\hat{K}_{21}(u)| ^ {a.s.} = O(1)\); and
- \(\sup_{u \in U} \left| \sum_{j=1}^{m} (\hat{K}_{21}(u) - K_{21}(u)) \right| ^ {a.s.} = O(1)\).

**Proof:** With Lemmas 2.12, 2.14 and 2.15, we have

\[
\sup_{u \in U, j \in \mathbb{Z}} |K_{21}(u)| \leq \sup_{u \in U, j \in \mathbb{Z}} |F_{R}(u)^{-1}| - p^2 \sup_{u \in U, j \in \mathbb{Z}} |F_{R}(u)^{-1}| \sup_{u \in U, j \in \mathbb{Z}} |z_i(u) z_i(u)^T| \sup_{u \in U, j \in \mathbb{Z}} |F_{R}(u)^{-1}| ^ {a.s.} = O(1).
\]

and

\[
\sup_{u \in U} \left| \sum_{j=1}^{m} (\hat{K}_{21}(u) - K_{21}(u)) \right| \leq \sup_{u \in U, j \in \mathbb{Z}} |z_i(u) \hat{z}_i(u)| \left( \sup_{u \in U, j \in \mathbb{Z}} |F_{R}(u)^{-1}| + \sup_{u \in U, j \in \mathbb{Z}} |F_{R}(u)^{-1}| \sup_{u \in U} \left| \sum_{j=1}^{m} (\hat{F}_{R}(u) - F_{R}(u)) \right| \right) + \sup_{u \in U, j \in \mathbb{Z}} |\hat{F}_{R}(u)| \sup_{u \in U} |F_{R}(u)^{-1}| \sup_{u \in U} \left| \sum_{j=1}^{m} (\hat{z}_i(u) \hat{z}_i(u)^T - z_i(u) z_i(u)^T) \right| ^ {a.s.} = O(1).
\]

**Lemma 2.19.** Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have

\[
\sup_{u \in U, j \in \mathbb{Z}} \left| \frac{\partial \text{vec}(F_{Q_i}(u))}{\partial u_2} \right| ^ {a.s.} = O(1) \quad \text{and} \quad \sup_{u \in U} \left| \sum_{j=1}^{m} \left( \frac{\partial \text{vec}(F_{Q_i}(u))}{\partial u_2} - \frac{\partial \text{vec}(\hat{F}_{Q_i}(u))}{\partial u_2} \right) \right| ^ {a.s.} = O(1).
\]
Proof: With \( \sup_{u \in U} |F_Q(u)|^{a.s.} \leq 1 \), we have

- \( \sup_{u \in U, r \in \mathbb{Z}} \left| \frac{\partial \text{vech} (F_{Q}(u))'}{\partial a} \right| = \frac{1 - a - b}{1 - b} = O(1); \)
- \( \sup_{u \in U, r \in \mathbb{Z}} \left| \frac{\partial \text{vech} (F_{Q}(u))'}{\partial a} \right| \leq \frac{1}{1 - \rho} + \sup_{u \in U} \left| \sum_{n=0}^{\infty} \rho^n z_{n-1}(u)z_{n-1}(u)' \right|^{a.s.} = O(1); \)
- \( \sup_{u \in U, r \in \mathbb{Z}} \left| \frac{\partial \text{vech} (F_{Q}(u))'}{\partial b} \right| \leq \frac{1}{(1 - \rho)^2} + \sup_{u \in U} \left| \sum_{n=0}^{\infty} np^{n-1} z_{n-1}(u)z_{n-1}(u)' \right|^{a.s.} = O(1), \)

where the validity of the second and third statement is implied by Lemma 2.2 and by Lemma 2.24 that is moved to Section 2.2.1 for reasons of clarity. Thus, the first part of the lemma holds with Lemmas 2.7 and 2.15. Furthermore, with Lemmas 2.6 and 2.15, we have

\[
\begin{align*}
\sup_{u \in U} \left\| \sum_{i=1}^{m} \left( \frac{\partial \text{vech} (F_{Q}(u))'}{\partial a} \right) \right\| = 0 \\
\sup_{u \in U} \left\| \sum_{i=1}^{m} \left( \frac{\partial \text{vech} (F_{Q}(u))'}{\partial b} \right) \right\| = \sup_{u \in U} \left\| \sum_{i=1}^{m} \sum_{k=0}^{\infty} b^k (z_{i-k} - \bar{z}_{i-k}) \right\|^{a.s.} = O(1)
\end{align*}
\]

Lemma 2.20. Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have

- \( \sup_{u \in U} \left\| \sum_{i=1}^{m} (F_{Q_i}(u) \otimes F_{Q_i}(u) - \bar{F}_{Q_i}(u) \otimes \bar{F}_{Q_i}(u)) \right\|^{a.s.} = O(1); \) and

- \( \sup_{u \in U} \left\| \sum_{i=1}^{m} (F_{Q_i}(u)F_{Q_i}(u) - \bar{F}_{Q_i}(u) \bar{F}_{Q_i}(u)) \right\|^{a.s.} = O(1). \)

Proof: Lemmas 2.6 and 2.7 and the mean value theorem yield

\[
\begin{align*}
\sup_{u \in U} \left\| \sum_{i=1}^{m} (F_{Q_i}(u) \otimes F_{Q_i}(u) - \bar{F}_{Q_i}(u) \otimes \bar{F}_{Q_i}(u)) \right\| \\
= \sup \max_{u \in U, 1 \leq i, j \leq p} \left| \sum_{i=1}^{m} \left[ |F_{Q_i}(u)|^{1/2} |F_{Q_i}(u)|^{-1/2} - |F_{Q_i}(u)|^{1/2} \bar{F}_{Q_i}(u) \right]_{ij} \right| \\
\leq 2 \sup \max_{u \in U, 1 \leq i, j \leq p} \left| |F_{Q_i}(u)|^{1/2} \right| \sup \max_{u \in U, 1 \leq i, j \leq p} \left| \sum_{i=1}^{m} \left[ |F_{Q_i}(u)|^{1/2} - |F_{Q_i}(u)|^{1/2} \right]_{ij} \right|
\end{align*}
\]
Proof: Furthermore, the mean value theorem and Lemma 2.6 yield
\[
\begin{align*}
& \leq 2 \sup_{u \in U} \left| \min_{1 \leq j \leq p} \left[ F_{Q}(u) \right]_{jj} \right|^{-1/2} \frac{1}{2} \sup_{1 \leq j \leq p} \left| \min_{1 \leq j \leq p} \left[ F_{Q}(u) \right]_{jj} \right|^{-3/2} \sup_{u \in U} \sum_{t=1}^{m} \left( F_{Q}(u) - \tilde{F}_{Q}(u) \right) \\
& \leq \left( \frac{1 - \frac{u}{1 - \rho} \right)^{2} \sup_{u \in U} \left| \sum_{t=1}^{m} \left( F_{Q}(u) - \tilde{F}_{Q}(u) \right) \right| \overset{a.s.}{=} O(1)
\end{align*}
\]
and analogously $\tilde{K}_{3}(u)$ in dependence of a finite past of observations.

Lemma 2.21. Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have

- $\sup_{u \in U, r \in \mathbb{Z}} \left| K_{3}(u) \right| \overset{a.s.}{=} O(1)$,
- $\sup_{u \in U, r \in \mathbb{Z}} \left| \tilde{K}_{3}(u) \right| \overset{a.s.}{=} O(1)$; and
- $\sup_{u \in U} \left| \sum_{t=1}^{m} \left( \tilde{K}_{3}(u) - K_{3}(u) \right) \right| \overset{a.s.}{=} O(1)$.

Proof: First, we have
\[
\begin{align*}
& \sup_{u \in U, r \in \mathbb{Z}} \left| \frac{\partial \text{vech} \left( F_{Q}(u) \right)}{\partial \text{vech} \left( F_{Q}(u) \right)} \right| \leq \frac{1}{2} \sup_{u \in U, r \in \mathbb{Z}} \left( \min_{1 \leq j \leq p} \left| F_{Q}(u) \right|_{jj} \right)^{-3/2} \overset{a.s.}{=} \left( \frac{1 - \frac{u}{1 - \rho} \right)^{3/2} = O(1). \tag{2.20}
\end{align*}
\]

Furthermore, the mean value theorem and Lemma 2.6 yield
\[
\begin{align*}
& \leq \frac{3}{4} \sup_{u \in U, r \in \mathbb{Z}} \left( \min_{1 \leq j \leq p} \left| F_{Q}(u) \right|_{jj} \right)^{-1/4} \sup_{u \in U} \sum_{t=1}^{m} \left( F_{Q}(u) - \tilde{F}_{Q}(u) \right) \leq \frac{3}{4} \left( \frac{1 - \frac{u}{1 - \rho} \right)^{3/2} \sup_{u \in U} \sum_{t=1}^{m} \left( F_{Q}(u) - \tilde{F}_{Q}(u) \right) \overset{a.s.}{=} O(1) \tag{2.21}
\end{align*}
\]
Lemma 2.23. Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 4.1, 4.2, we have

- \[ \sup_{u \in U, t \in \mathbb{R}} \left| \frac{\partial \text{vech}(F_{Q_t}(u))}{\partial u_2} \right| \overset{a.s.}{=} O(1), \]
- \[ \sup_{u \in U, t \in \mathbb{R}} \left| \frac{\partial \text{vech}(F_{Q_t}(u))}{\partial u_2} \right| \overset{a.s.}{=} O(1); \text{ and} \]
- \[ \sup_{u \in U} \left| \sum_{t=1}^{m} \left( \frac{\partial \text{vech}(F_{Q_t}(u))}{\partial u_2} - \frac{\partial \text{vech}(F_{Q_t}(u))}{\partial u_2} \right) \right| \overset{a.s.}{=} O(1). \]

Proof: With Lemmas 2.19 and 2.21, we have

\[ \sup_{u \in U, t \in \mathbb{R}} \left| \frac{\partial \text{vech}(F_{R_t}(u))}{\partial u_2} \right| \leq \sup_{u \in U, t \in \mathbb{R}} |K_{3t}(u)| \sup_{u \in U, t \in \mathbb{R}} \left| \frac{\partial \text{vech}(F_{Q_t}(u))}{\partial u_2} \right| \overset{a.s.}{=} O(1) \]

and

\[ \sup_{u \in U} \left| \sum_{t=1}^{m} \left( \frac{\partial \text{vech}(F_{R_t}(u))}{\partial u_2} - \frac{\partial \text{vech}(F_{R_t}(u))}{\partial u_2} \right) \right| \leq \sup_{u \in U, t \in \mathbb{R}} |K_{3t}(u)| \sup_{u \in U} \left| \sum_{t=1}^{m} \left( \frac{\partial \text{vech}(F_{Q_t}(u))}{\partial u_2} - \frac{\partial \text{vech}(F_{Q_t}(u))}{\partial u_2} \right) \right| \overset{a.s.}{=} O(1). \]
Proof: With Lemmas 2.18 and 2.19, we have

\[
\sup_{t \in U, j \in Z} \left| \frac{\partial l_t(u)}{\partial u} \right| \leq \sup_{t \in U, j \in Z} \left| \frac{\partial \text{vec}(F_{R_j}(u))'}{\partial u} \right| + \sup_{t \in U, j \in Z} |K_{2j}(u)|
\]

and

\[
\sup_{t \in U} \sum_{j=1}^m \left| \frac{\partial l_t(u)}{\partial u} - \frac{\partial l_t(u)}{\partial u} \right| \leq \sup_{t \in U} |K_{2j}(u)| \sup_{t \in U} \left| \sum_{j=1}^m \left( \frac{\partial \text{vec}(F_{R_j}(u))'}{\partial u} - \frac{\partial \text{vec}(F_{R_j}(u))'}{\partial u} \right) \right|
\]

\[
+ \sup_{t \in U, j \in Z} \left| \frac{\partial \text{vec}(F_{R_j}(u))'}{\partial u} \right| \sup_{t \in U} \left| \sum_{j=1}^m \left( F_{R_j}(u)^{-1} - \bar{F}_{R_j}(u)^{-1} \right) \right|
\]

\[
+ \sup_{t \in U, j \in Z} \left| \frac{\partial \text{vec}(F_{R_j}(u))'}{\partial u} \right| \sup_{t \in U} \left| \sum_{j=1}^m \left( \bar{K}_{2j}(u) - K_{2j}(u) \right) \right|^{\frac{1}{2}} = O(1).
\]

With Lemmas 2.17 and 2.23, we have \( \sup_{t \in U} \left| \bar{D}_m(u) - D_m(u) \right|^{\frac{1}{2}} = O(\frac{1}{m}) \), which completes the proof of (2.3).

\( \blacksquare \)

2.1.2 The Proof of \( \mathbb{E} \sup_{u \in U} \left| l'_0(u)l'_0(u)^T \right| < \infty \)

Along the lines of Berkes et al. (2003), Lemmas 2.17 and 2.23 yield

\[
\sup_{u \in U} \left| l'_0(u)l'_0(u)^T \right| \leq \left( \sup_{u \in U} \left| \frac{\partial l_0(u)}{\partial u} \right| \right)^2 \overset{a.s.}{=} O(1)
\]

This implies

\[
\mathbb{E} \left[ \sup_{u \in U} \left| l'_0(u)l'_0(u)^T \right| \right] < \infty.
\]

2.1.3 The Proof of the Uniform Convergence of \( D_m(\cdot) \) to \( D(\cdot) \)

We use Theorem A.2.2 in White (1994). The conditions are satisfied, since \( U \) is a compact set and \( l'_t(u)l'_t(u)^T \) is ergodic, continuous in \( u \) for all \( y_t \) and measurable in \( y_t \) for all \( u \in U \). We choose the dominating function as \( \sup_{u \in U} \left| l'_t(u)l'_t(u)^T \right| \). Thus, the finiteness of the expectation is implied by (2.4), which yields the uniform convergence of \( D_m(\cdot) \) to \( D(\cdot) \).

\( \blacksquare \)

2.1.4 The Proof the Proposition.

Since \( D_m(\cdot) \) converges uniformly to \( D(\cdot) \), (2.1) follows directly from the consistency of the estimator \( \hat{\theta}_m \) and the positive definiteness of the variation matrix \( D \), that is from Assumption 2.1, which completes the proof of Proposition 4.1.

\( \blacksquare \)
2.2 The Proof that the Second Order Partial Derivatives are of Finite Expectation

2.2.1 Notation

Throughout the next section, for \( \varphi \in (0, 1) \) and \( i, j \in \{1, \ldots, p\} \), we use the notation

- \( G_0^y(i, j, \varphi) := \sum_{n=0}^{\infty} \varphi^n y_{i,n-1} y_{j,n-1} \)
- \( G_1^y(i, j, \varphi) := \sum_{n=0}^{\infty} n \varphi^{n-1} y_{i,n-1} y_{j,n-1} \) \hspace{1cm} (2.22)
- \( G_2^y(i, j, \varphi) := \sum_{n=0}^{\infty} n(n-1) \varphi^{n-2} y_{i,n-1} y_{j,n-1} \) \hspace{1cm} (2.23)
- \( G_0^z(i, j, u) := \sum_{n=0}^{\infty} b^n z_{i,n-1}(u) z_{j,n-1}(u) \)
- \( G_1^z(i, j, u) := \sum_{n=0}^{\infty} n b^{n-1} z_{i,n-1}(u) z_{j,n-1}(u) \)
- \( Z(u) := \left( \sum_{s=1}^{p} Z_{s0}(u) \right)^2 = \sum_{s=1}^{p} \sum_{s=1}^{p} Z_{s0}(u) Z_{s0}(u). \) \hspace{1cm} (2.24)

Note that \( G^y(i, j, u) \leq \frac{1}{c_i} G^y(i, j, b) \) for all \( u \in U \). The existence of the expectation of some products of the terms (2.22)-(2.26) can be obtained by the use of the following lemma.

**Lemma 2.24.** Let \( G_1, G_2, G_3 \) and \( G_4 \) be arbitrary terms from (2.22)-(2.26) for any indices \( i, j \in \{1, \ldots, p\} \) and parameter values \( \varphi \in (0, 1) \) or \( u \in U \). We have

\[
E \left( \prod_{i=1}^{k} G_i \right) < \infty, \quad \text{for } i \in \{1, \ldots, 4\}.
\]

**Proof:** The proof can be demonstrated for four terms of type (2.22) and be extended to the remaining cases. Define \( G_k := G_0^y(i_k, j_k, \varphi_k) \) for \( k = 1, \ldots, 4 \) and \( \varphi := \max \{\varphi_1, \ldots, \varphi_4\} \). First, we have for \( \varphi_1, \varphi_2 \in (0, 1) \) and all combinations of \( i, j, q, r \in \{1, \ldots, p\} \) and by the use of the Cauchy product rule:

\[
G_0^y(i, j, \varphi_1) G_0^y(q, r, \varphi_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varphi_1^n \varphi_2^m y_{i,n-1} y_{j,n-1} y_{q,m-1} y_{r,m-1} \leq \sum_{m=0}^{\infty} \left( \varphi_1 \right)^m \sum_{n=0}^{\infty} \left( \varphi_2 \right)^n y_{i,n-1} y_{j,n-1} y_{q,m-1} y_{r,m-1}. \] \hspace{1cm} (2.27)

With Assumptions 3.5 and 3.6 the double sum in (2.27) is stochastically bounded, since by the use of the Cauchy Schwarz inequality, we have

\[
E|y_{is} y_{js}| \leq \left[ E \left( y_{is}^2 \right) \right]^{1/2} \left[ E \left( y_{js}^2 \right) \right]^{1/2} < \infty, \quad \forall \ s, t \in \mathbb{Z} \quad \text{and} \quad i, j, k, l \in \{1, \ldots, p\}.
\]
Analogously, all products \(G_{m_1}^i(i, j, \varphi_1)G_{m_2}^j(k, l, \varphi_2)\) or \(G_{m_1}^i(i, j, u)G_{m_2}^j(k, l, u)\) with \(\varphi_1, \varphi_2 \in (0, 1)\) and \(m_1, m_2 \in \{0, 1, 2\}\) are of finite expectation. This property can be extended to products of up to four terms as in (2.22)-(2.26), which can be shown by the use of Assumptions 3.5 and 3.6, the repeated utilization of the Cauchy product rule and the application of the generalized Hölder inequality, that is Lemma 1.16 in Alt (2006) with \(m = 8, q = 1\) and \(p_i = 8, i = 1, \ldots, 8\). ■

### 2.2.2 The Partial Derivatives of \(w_{i0}(u)\)

The following statements on the partial derivatives of \(w_{i0}(u)\) with respect to the variance parameters will be used in the next sections:

\[
\left(\frac{\partial w_{i0}(u)}{\partial x_i}\right)^2 = \frac{1}{4} \frac{1}{(1-t_i)^2} \quad \left(\frac{\partial w_{i0}(u)}{\partial s_i}\right)^2 = \frac{1}{4} G_0^y(i, i, t_i)^2
\]  

\[
\left(\frac{\partial w_{i0}(u)}{\partial t_i}\right)^2 = \frac{1}{4} \frac{x_i^2}{(1-t_i)^2} + \frac{1}{2} \frac{x_i}{(1-t_i)^2} G_1^y(i, i, t_i) + \frac{1}{4} G_1^y(i, i, t_i)^2
\]  

\[
\frac{\partial^2 w_{i0}(u)}{(\partial x_i)^2} \leq \frac{1}{4 C_1^{3/2} (1-t_i)^2} \quad \frac{\partial^2 w_{i0}(u)}{(\partial s_j)^2} \leq \frac{G_1^y(j, j, t_j)}{2 C_1^{1/2}} \quad \frac{\partial^2 w_{i0}(u)}{(\partial t_i)^2} \leq \frac{G_1^y(j, j, t_j)}{2 C_1^{1/2}}
\]

\[
\frac{\partial^2 w_{i0}(u)}{\partial x_i \partial s_j} \leq \frac{G_1^y(j, j, t_j)}{2 C_1^{3/2}} \quad \frac{\partial^2 w_{i0}(u)}{\partial x_i \partial t_j} \leq \frac{G_1^y(j, j, t_j)}{2 C_1^{3/2} (1-t_i)} \quad \frac{\partial^2 w_{i0}(u)}{\partial s_j \partial t_j} \leq \frac{G_1^y(j, j, t_j)^2}{4 C_1^{3/2}} + \frac{G_1^y(j, j, t_j)}{2 C_1^{1/2}}
\]  

Furthermore, the following statements on derivatives of the roots of \(w_{i0}(u)\) will be useful:

\[
\frac{\partial w_{i0}(u)^{1/2}}{\partial x_j} \leq \frac{1}{2 C_1^{1/2} (1-t_j)} \quad \frac{\partial w_{i0}(u)^{1/2}}{\partial s_j} \leq \frac{G_1^y(j, j, t_j)}{2 C_1^{1/2}} \quad \frac{\partial w_{i0}(u)^{1/2}}{\partial t_j} \leq \frac{G_1^y(j, j, t_j)}{2 C_1^{1/2}}
\]  

\[
\frac{\partial^2 w_{i0}(u)^{1/2}}{(\partial x_i)^2} \leq \frac{1}{4 C_1^{3/2} (1-t_i)^2} \quad \frac{\partial^2 w_{i0}(u)^{1/2}}{(\partial s_j)^2} \leq \frac{G_1^y(j, j, t_j)}{2 C_1^{1/2}} \quad \frac{\partial^2 w_{i0}(u)^{1/2}}{(\partial t_i)^2} \leq \frac{G_1^y(j, j, t_j)}{2 C_1^{1/2}}
\]  

\[
\frac{\partial^2 w_{i0}(u)^{1/2}}{\partial x_i \partial s_j} \leq \frac{G_1^y(j, j, t_j)}{2 C_1^{3/2}} + \frac{G_1^y(j, j, t_j)}{2 C_1^{3/2} (1-t_i)} \quad \frac{\partial^2 w_{i0}(u)^{1/2}}{\partial s_j \partial t_j} \leq \frac{G_1^y(j, j, t_j)^2}{4 C_1^{3/2}} + \frac{G_1^y(j, j, t_j)}{2 C_1^{1/2}}
\]
2.2.3 The Partial Derivatives of $F_{Q_0}(u)$ and $F_{Q_0'}(u)$

Now, we have a closer look at the first and second order partial derivatives of the $(i, j)$-th entry of $F_{Q_0}(u)$.

\[
\begin{align*}
\frac{\partial [F_{Q_0}(u)]_{ij}}{\partial a} &= G_0^c(i, j, u) - \frac{1}{1 - b} \\
\frac{\partial^2 [F_{Q_0}(u)]_{ij}}{\partial (\partial a)^2} &= 0 \\
\frac{\partial [F_{Q_0}(u)]_{ij}}{\partial b} &= a G_1^c(i, j, u) - \frac{a}{(1 - b)^2} \\
\frac{\partial^2 [F_{Q_0}(u)]_{ij}}{\partial (\partial b)^2} &= a G_2^c(i, j, u) - \frac{2a}{(1 - b)^3}
\end{align*}
\]  

The entries of $F_{Q_0'}(u)$ are either 0 or $[F_{Q_0}(u)]_{ii}^{1/2}$, for $i \in \{1, \ldots, p\}$. Hence, only the derivatives of the main diagonal entries will be considered.

\[
\begin{align*}
\frac{\partial^2 [F_{Q_0'}(u)]_{ii}}{\partial (\partial a)^2} &= \frac{\partial^2 [F_{Q_0}(u)]_{ii}^{1/2}}{\partial (\partial a)^2} = \frac{3}{4} [F_{Q_0}(u)]_{ii}^{-3/2} \left( \frac{\partial [F_{Q_0}(u)]_{ii}}{\partial a} \right)^2 < \frac{3a^2}{4} \left( \frac{1 - b}{1 - a - b} \right)^{3/2} G_0^n(i, i, u)^2 \\
\frac{\partial^2 [F_{Q_0'}(u)]_{ii}}{\partial (\partial b)^2} &= \frac{3}{4} [F_{Q_0}(u)]_{ii}^{-3/2} \left( \frac{\partial [F_{Q_0}(u)]_{ii}}{\partial b} \right)^2 < \frac{3a^2}{4} \left( \frac{1 - b}{1 - a - b} \right)^{3/2} \frac{2}{(1 - \rho)^3} \\
\frac{\partial^2 [F_{Q_0'}(u)]_{ii}}{\partial (\partial a \partial b)} &= \frac{3}{4} [F_{Q_0}(u)]_{ii}^{-3/2} \frac{\partial [F_{Q_0}(u)]_{ii}}{\partial a} \frac{\partial [F_{Q_0}(u)]_{ii}}{\partial b} - \frac{1}{2} [F_{Q_0}(u)]_{ii}^{-3/2} \frac{\partial^2 [F_{Q_0}(u)]_{ii}}{\partial (\partial a \partial b)} < \frac{3a^2}{4} \left( \frac{1 - b}{1 - a - b} \right)^{3/2} G_0^n(i, i, u) G_1^n(i, i, u) + \frac{1}{2} \left( \frac{1 - b}{1 - a - b} \right)^{3/2} \frac{1}{(1 - \rho)^2}
\end{align*}
\]

2.2.4 The Proof that the Expectation of $\frac{\partial^2 \log u}{\partial u \partial u'}$ is Finite

For the expectation of the second order partial derivatives with respect to the variance parameters, we have

\[
E \left( \frac{\partial^2 \log u}{\partial u \partial u'} \right) = -\frac{1}{2} E \left( \frac{\partial \text{vec}(F_{D_0}(u))'}{\partial u} \right) \left[ 2 \left( F_{D_0}(u)^{-1} \otimes F_{D_0}(u)^{-1} \right) \\
+ \left( z_0(u)z_0'(u) \otimes F_{D_0}(u)^{-1} F_{D_0}(u)^{-1} F_{D_0}(u)^{-1} \right) \\
+ \left( F_{D_0}(u)^{-1} F_{D_0}(u)^{-1} F_{D_0}(u)^{-1} \otimes z_0(u)z_0'(u) \right) \right] \left( \frac{\partial \text{vec}(F_{D_0}(u))}{\partial u'} \right)
\]
\[ + \mathbb{E}\left(\frac{1}{2} \left( \text{vec} \left( F_{D_0}(u)^{-1} F_{R_0}(u)^{-1} z_0(u) z_0'(u) \right) \otimes I_p \right) \right) \]  
\[ + \frac{1}{2} \left( \text{vec} \left( z_0(u) z_0'(u) F_{R_0}(u)^{-1} F_{D_0}(u)^{-1} \right) \otimes I_p \right) \]  
\[ - \left( \text{vec} \left( F_{D_0}(u)^{-1} \right) \otimes I_p \right) \frac{\partial^2 \text{vec} \left( F_{D_0}(u) \right)^{\prime}}{\partial u \partial u'} \right). \]  

The expectations of the Kronecker products in (2.42)-(2.48) all possess a similar structure. By a propiate choice of the matrices

\[ V^{(l)} := \left[ v_{ij}^{(l)} \right]_{i,j=1,...,p}, \quad l \in \{1, 2\} \]

these terms can be split up into

\[ \frac{\partial \text{vec} \left( F_{D_0}(u) \right)^{\prime}}{\partial u_1} \left( V^{(1)} \otimes V^{(2)} \right) \frac{\partial \text{vec} \left( F_{D_0}(u) \right)}{\partial u_1^\prime}. \]  

Note that (2.52) is a block diagonal matrix with \((3 \times 3)\) blocks

\[ \frac{v_{kk}^{(1)} v_{kk}^{(2)} \partial w_{k0}(u) \partial w_{k0}(u)}{w_{k0}(u)} \frac{\partial r_{k0}(u)}{\partial r_k^*}, \quad k \in \{1, \ldots, p\} \]

on its main diagonal. For (2.42) the expression in (2.52) becomes

\[ \frac{v_{ii}^{(1)} v_{ii}^{(2)} w_{i0}(u)}{w_{i0}(u)} = w_{i0}(u)^{-2} \leq C_1^{-2}. \]

Thus, the expectation of this summand is finite.

In the following, denote \( F_{R_0}(u)^{-1} := \left[ r_{i,j0}(u) \right]_{i,j=1,...,p} \) and recall that the arguments in the proof of Lemma 2.14 imply

\[ \sup_{u \in U, t \in \mathbb{R}} \left| r_{i,j0}(u) \right| \leq \left\{ \frac{1 - b}{1 - a - b \delta_1} \right\}^p (p - 1)! =: \delta, \quad \forall i, j \in \{1, \ldots, p\}. \]  

Hence, for (2.43) and (2.44), we have

\[ \frac{v_{ii}^{(1)} v_{ii}^{(2)}}{w_{i0}(u)} \leq C_1^{-3} y_{i0}^2 \left| r_{ii0}(u) \right| \leq \delta, \]  

and for (2.45)-(2.48), we have

\[ \frac{v_{ii}^{(1)} v_{ii}^{(2)}}{w_{i0}(u)} = \frac{y_{i0}}{w_{i0}(u)^2} \sum_{k=1}^{p} \frac{y_{k0}}{w_{k0}(u)} \left| r_{i,k0}(u) \right| \leq \delta, \]  

Hence, with Lemma 2.24 and Assumptions 3.5 and 3.6 it is obvious that products of one of the terms (2.28)-(2.31) and (2.54) or (2.55) are of finite expectation.
Furthermore, note that the terms (2.49)-(2.51) are finite if this applies to the following expectations for all $u \in U$, $i_1, \ldots, i_4 \in \{1, \ldots, p\}$ and $x, y \in \{x_{i_4}, s_{i_4}, t_{i_4}\}$:

$$
\begin{align*}
E \left( w_{i_0}(u)^{-1/2} r_{i_0 i_0}(u) \right) & \leq \delta_s C_1^{-1/2} E \left( Z(u) \frac{\partial^2 w_{i_0}(u)^{1/2}}{\partial x \partial y} \right) \quad (2.56) \\
E \left( w_{i_0}(u)^{-1/2} \frac{\partial^2 w_{i_0}(u)^{1/2}}{\partial x \partial y} \right) & \leq C_1^{-1/2} E \left( \frac{\partial^2 w_{i_0}(u)^{1/2}}{\partial x \partial y} \right). \quad (2.57)
\end{align*}
$$

The finity of (2.56) and (2.57) is a direct consequence of Lemma 2.24, (2.32)-(2.35) and (2.53). Therefore, combining the previous results finally yields

$$
E \left( \frac{\partial^2 l_0(u)}{\partial u_1 \partial u'_1} \right) < \infty. \quad \blacksquare
$$

### 2.2.5 The Proof that the Expectation of $\frac{\partial^2 l_0(u)}{\partial u_1 \partial u'_2}$ and $\frac{\partial^2 l_0(u)}{\partial u_2 \partial u'_1}$ is Finite

For the expectation of the cross derivatives of the gradient contributions with respect to the variance and correlation parameters, we have

$$
E \left( \frac{\partial^2 l_0(u)}{\partial u_1 \partial u'_2} \right) = -\frac{1}{4} E \left( \frac{\partial \text{vec} \left( F_{D_0}(u) \right)}{\partial u_1} \right) \left[ (z_0(u)z_0(u)^\prime F_{R_0}(u)^{-1} \otimes F_{D_0}(u)^{-1} F_{R_0}(u)^{-1}) + (F_{D_0}(u)^{-1} F_{R_0}(u)^{-1} \otimes z_0(u)z_0(u)^\prime F_{R_0}(u)^{-1}) \right] 2 \left( D_{p, q}^+ \right)^\prime \frac{\partial \text{vec}(F_{R_0}(u))}{\partial u'_2} \quad (2.58)
$$

The cross derivatives are of finite expectation, if the following expectations are finite for all $u \in U$, $i_1, \ldots, i_5 \in \{1, \ldots, p\}$, $x \in \{x_{i_4}, s_{i_4}, t_{i_4}\}$ and $y \in \{a, b, q_1, \ldots, q_{p'}\}$:

$$
\begin{align*}
E \left( w_{i_0}(u)^{-1/2} r_{i_0 i_0}(u) \right) & \leq \delta_s C_1^{-1/2} E \left( Z(u) \frac{\partial^2 w_{i_0}(u)^{1/2}}{\partial x \partial y} \right) \quad (2.59) \\
E \left( w_{i_0}(u)^{-1/2} \frac{\partial^2 w_{i_0}(u)^{1/2}}{\partial x \partial y} \right) & \leq \frac{1}{C_1^{-1/2}} E \left( \frac{\partial^2 w_{i_0}(u)^{1/2}}{\partial x \partial y} \right). \quad (2.60)
\end{align*}
$$

The finity of (2.59) and (2.60) is implied by Lemma 2.24, (2.53) and the statements in Section 2.2.2 and 2.2.3, which completes the proof.
2.2.6 The Proof that the Expectation of $\frac{\partial^2 h(u)}{\partial u_2 \partial u_2}$ is Finite

First, it has to be shown that the following expectations are finite:

$$E\left( \frac{\partial \text{vec}(F_{\mathcal{R}_0}(u))}{\partial u_2} \right)^T \left( F_{\mathcal{R}_0}(u)^{-1} \otimes F_{\mathcal{R}_0}(u)^{-1} \right) \frac{\partial \text{vec}(F_{\mathcal{R}_0}(u))}{\partial u_2}$$ (2.61)

$$E\left( \frac{\partial \text{vec}(F_{\mathcal{R}_0}(u))}{\partial u_2} \right)^T \left( F_{\mathcal{R}_0}(u)^{-1} \otimes F_{\mathcal{R}_0}(u)^{-1} \right) \frac{\partial \text{vec}(F_{\mathcal{R}_0}(u))}{\partial u_2}$$ (2.62)

$$E\left( \frac{\partial \text{vec}(F_{\mathcal{R}_0}(u))}{\partial u_2} \right)^T \left( F_{\mathcal{R}_0}(u)^{-1} \otimes F_{\mathcal{R}_0}(u)^{-1} \right) \frac{\partial \text{vec}(F_{\mathcal{R}_0}(u))}{\partial u_2}$$ (2.63)

The following statements will be useful for the next parts of this section:

- $d(i, j) := \left( [F_{\mathcal{Q}_0}(u)]_{ii} [F_{\mathcal{Q}_0}(u)]_{jj} \right)^{-1/2}$ a.s. $\leq \frac{1 - b}{1 - a - b}$

- $G_0^2(k, k, u) \leq \frac{G_0^2(k, k, u)}{aG_0^2(k, k, u)} = \frac{1}{a}$

- $[F_{\mathcal{Q}_0}(u)]_{ij} d(i, j) \leq 1 \Rightarrow \frac{[F_{\mathcal{Q}_0}(u)]_{ij}}{[F_{\mathcal{Q}_0}(u)]_{ii}^{1/2}} \leq \frac{[F_{\mathcal{Q}_0}(u)]_{jj}^{1/2}}{[F_{\mathcal{Q}_0}(u)]_{ii}^{1/2}} \leq \frac{1}{1 + aG_0^2(j, j, u)}$

The finity of (2.62) and (2.63) is a direct consequence of the finity of the following expectations for $u \in U$,

$$E\left( \left[ \frac{\partial^2 h(u)}{\partial x^2} \right]_{ii} \right)^2 \leq \delta^2 E\left( \left[ \frac{\partial^2 h(u)}{\partial x^2} \right]_{ii} \right)^2 \leq \delta^2 \left( 1 + aG_0^2(i, i, u) \right) Z(u)$$ (2.68)
Finally, with the finiteness of the terms of the right-hand sides of (2.70) is a consequence of (2.53) and Lemma 2.24. Analogously, the finiteness of (2.61) is obtained by replacing the terms (2.66) and (2.67) are finite if the same applies to the following expectations for all \(i, \ldots, i_5 \in \{1, \ldots, p\}\):

\[
E\left[ r_{i_1 i_2 0}(u) Z(u) \frac{\partial [F_{Q_0}(u)]_{i_1 i_3}}{\partial x} \frac{\partial [F_{Q_0}(u)]_{i_2 i_3}}{\partial y} \frac{\partial^2 [F_{Q_0}(u)]_{i_1 i_3}}{\partial x \partial y} \right] \\
\leq \delta^2 \left( \frac{1 - b}{1 - a - b} \right)^{1/2} E\left[ Z(u) \frac{\partial [F_{Q_0}(u)]_{i_1 i_3}}{\partial x} \right] \leq \delta^2 \left( \frac{1 - b}{1 - a - b} \right)^{1/2} E\left[ Z(u) \right] \leq \delta^2 \left( \frac{1 - b}{1 - a - b} \right)^{1/2} E\left[ \frac{\partial [F_{Q_0}(u)]_{i_1 i_3}}{\partial x} \right] \leq \delta^2 \left( \frac{1 - b}{1 - a - b} \right)^{1/2} E\left[ Z(u) \frac{\partial^2 [F_{Q_0}(u)]_{i_1 i_3}}{\partial x \partial y} \right].
\]

The finiteness of the terms of the right-hand sides of (2.70)-(2.70) is a consequence of (2.53) and Lemma 2.24. Finally, with

\[
\frac{\partial}{\partial u_2} \text{vec} \left( \frac{\partial \text{vech} \left( F_{Q_0}(u) \right)}{\partial u_2} \right) = \begin{bmatrix}
\frac{\partial^2 \text{vech} \left( F_{Q_0}(u) \right)}{\partial u_2^2}
\frac{\partial^2 \text{vech} \left( F_{Q_0}(u) \right)}{\partial u_2 \partial x}
\frac{\partial^2 \text{vech} \left( F_{Q_0}(u) \right)}{\partial u_2 \partial y}
\frac{\partial^2 \text{vech} \left( F_{Q_0}(u) \right)}{\partial u_2 \partial y}
\end{bmatrix}
\begin{bmatrix}
0_{p^+ \times p^-}
0_{p^+ \times p^-}
0_{p^+ \times p^-}
0_{p^+ \times p^-}
\end{bmatrix},
\]

the terms (2.66) and (2.67) are finite if the same applies to the following expectations for all \(i, \ldots, i_5 \in \{1, \ldots, p\}\):

\[
E\left[ r_{i_1 i_2 0}(u) Z(u) \frac{\partial^2 [F_{Q_0}(u)]_{i_1 i_3}}{\partial x \partial y} \right] \leq \delta^2 \left( \frac{1 - b}{1 - a - b} \right) E\left[ Z(u) \frac{\partial^2 [F_{Q_0}(u)]_{i_1 i_3}}{\partial x \partial y} \right].
\]
2. THE PROOFS OF THE THEORETICAL STATEMENTS

\[ \mathbb{E}\left[ r_{i_2,0}(u) \right]^2 Z(u) \frac{[F_{\Theta}(u)]_{i_3} \frac{\partial^2 [F_{\Theta}(u)]_{i_6}}{\partial x \partial y}}{[F_{\Theta}(u)]_{i_5}} \leq \delta^2 \mathbb{E}\left[ Z(u) \left[ 1 + aG_0(i_4, i_4, u) \right]^{1/2} \frac{\partial^2 [F_{\Theta}(u)]_{i_6}}{\partial x \partial y} \right]. \] (2.74)

With Lemma 2.24, (2.53) and the statements from Section 2.2.3, the terms (2.73) and (2.74) are finite which completes the proof and yields

\[ \mathbb{E}\left[ \frac{\partial^2 l_0(u)}{\partial u_2 \partial u_2'} \right] < \infty. \]  

\[ \text{\Halmos} \]

2.3 The Proof of Theorem 4.1

The proof follows the arguments of the proof of Theorem 3.1 in Berkes et al. (2004). It is shown that the process \( \{ \hat{l}_t(\hat{\Theta}_m), t \in \mathbb{Z} \} \) behaves similarly as \( \{ l_t(\theta), t \in \mathbb{Z} \} \). This is a stationary and ergodic martingale difference sequence, which yields the convergence to a Gaussian limit process. The approach will be sketched in this section.

First, we have for \( m \to \infty \) the following statement that equals Lemma 6.3 in Berkes et al. (2004):

\[ \sup_{1 < k < \infty} \frac{\sum_{i=m+1}^{m+k} \hat{l}_i(\hat{\Theta}_m) - \sum_{i=m+1}^{m+k} l_i(\theta) + (\hat{\Theta}_m - \theta) k \mathbb{E}\left[ l''(\theta) \right]}{\sqrt{m} \left( 1 + \frac{k}{m} \right) b \left( \frac{k}{m} \right)} = o_P(1). \] (2.75)

The validity of (2.75) is implied by the arguments in the proof of Proposition 4.1 and the uniform convergence \( l''(\cdot) \) to \( \mathbb{E}\left[ l''(\cdot) \right] \). The latter one is implied by Theorem A.2.2 in White (1994). Since \( U \) is a compact set and \( l''(u) \) is continuous in \( u \) for all \( y_t \) as well as measurable in \( y_t \) for all \( u \in U \), the dominance condition remains to be verified. We choose the dominating function as \( \sup_{u \in U} l''(u) \). Hence, the results of Section 2.2 yield

\[ \mathbb{E}\left[ \sup_{u \in U} l''(u) \right] < \infty. \]

Therefore, Theorem A.2.2 in White (1994), implies

\[ \sup_{u \in U} \left| \frac{1}{m} \sum_{i=1}^{m} l''(u) - \mathbb{E}\left[ l''(u) \right] \right| = \frac{1}{m} \sup_{u \in U} \left| \sum_{i=1}^{m} (l''(u) - \mathbb{E}\left[ l''(u) \right]) \right| \xrightarrow{a.s.} 0. \]
Analogously to Lemma 6.4 in Berkes et al. (2004) and by the use of Assumptions 3.7 and 3.8, we have for $m \to \infty$:

$$
\left( \hat{\theta}_m - \theta \right) = \frac{1}{m} \sum_{i=1}^{m} l_i' \left( \theta \right) \left[ E(l_0' \left( \theta \right)) \right]^{-1} \left[ 1 + o(1) \right].
$$

Furthermore, Assumptions 3.7, 3.8 and 4.3 and Lemma 6.5 in Berkes et al. (2004) yield

$$
\sup_{1 \leq k < \infty} \frac{\left| \sum_{i=m+1}^{m+k} l_i' \left( \hat{\theta}_m \right) - \left[ \sum_{i=m+1}^{m+k} l_i' \left( \theta \right) - \frac{k}{m} \sum_{i=1}^{m} l_i' \left( \theta \right) \right] \right|}{\sqrt{m} \left( 1 + \frac{k}{m} \right) b \left( \frac{k}{m} \right)} = o_P(1).
$$

Then, by the use of Lemma 6.6 in Berkes et al. (2004) we have:

$$
\sup_{1 \leq k < \infty} \frac{\left| \sum_{i=m+1}^{m+k} l_i' \left( \theta \right) - \frac{k}{m} \sum_{i=1}^{m} l_i' \left( \theta \right) \right|}{\sqrt{m} \left( 1 + \frac{k}{m} \right) b \left( \frac{k}{m} \right)} = \sup_{0 < t < \infty} \frac{|W_D(1 + t) - (1 + t)W_D(1)|}{(1 + t)b(t)}.
$$

(2.76)

where $\{W_D(t), t \in [0, B]\}$ is a $d$-variate Gaussian process with

$$
E \left[ W_D(s) \right] = 0_d, \quad \forall \ s \in [0, B), \quad \text{and} \quad E \left[ W_D^T(k)W_D(l) \right] = \min \{k, l\} D, \quad \forall \ k, l \in [0, B].
$$

Moreover, the Lemmas 2.17 and 2.23 imply

$$
\sup_{u \in U} \left| \sum_{i=1}^{n} \left( \hat{l}_i(u) - l_i(u) \right) \right| \overset{d.s.}{=} O(1).
$$

(2.77)

As a consequence of (2.76) and (2.77) we have

$$
\sup_{1 \leq k < mB} \frac{\left| \sum_{i=m+1}^{m+k} D_m^{-\frac{1}{2}} \hat{l}_i \left( \hat{\theta}_m \right) \right|}{m^{\frac{1}{2}} \left( 1 + \frac{k}{m} \right) b \left( \frac{k}{m} \right)} = \sup_{0 < t \leq B} \frac{D^{-\frac{1}{2}} \left| W_D(1 + t) - (1 + t)W_D(1) \right|}{(1 + t)b(t)}.
$$

A simple recalculation of the properties of the resulting process indicates that

$$
\left\{ D^{-\frac{1}{2}} \left| W_D(1 + t) - (1 + t)W_D(1) \right|, t \in [0, \infty) \right\}
$$

and $\{G(t), t \in [0, \infty)\}$ possess the same distribution.
2.4 The Proof of Theorem 4.2

Under the alternative of a change in the vector of parameters, it may be appropriate to decompose the detector as in Berkes et al. (2004):

\[
\frac{\sum_{i=m+1}^{m+k} l'_i(\hat{\theta}_m)}{\sqrt{m} \left(1 + \frac{k}{m}\right) b \left(\frac{k}{m}\right)} = \frac{\sum_{i=m+1}^{m+k-1} l'_i(\hat{\theta}_m)}{\sqrt{m} \left(1 + \frac{k}{m}\right) b \left(\frac{k}{m}\right)} + \frac{m+k}{\sqrt{m} \left(1 + \frac{k}{m}\right) b \left(\frac{k}{m}\right)}. \tag{2.78}
\]

The first summand on the righthand side of (2.78) is based on the gradient contributions of observations before the parameter change. It can be treated analogously to the proof of Theorem 4.1 or to the proof of Theorem 3.1. in Berkes et al. (2004). Note that \( m \to \infty \) implies \( k^* \to \infty \) and we have

\[
\sup_{k^* \leq k < \infty} \left| \frac{\sum_{i=m+1}^{m+k} l'_i(\hat{\theta}_m)}{\sqrt{m} \left(1 + \frac{k}{m}\right) b \left(\frac{k}{m}\right)} \right| \overset{d}{\to} \sup_{t \in [\lambda^* B, \infty)} \frac{|W_D (1 + \lambda^* B) - (1 + \lambda^* B) W_D(1)|}{(1 + r) b (t)}. \tag{2.79}
\]

The second summand on the righthand side of (2.78) is based on the gradient contributions of the observations after the change which are driven by the parameter vector \( \theta^* \). Thus for \( m \to \infty \), a Taylor series expansion of \( l''_i(\hat{\theta}_m) \) centered in \( \theta^* \) yields

\[
\sup_{k^* \leq k < \infty} \left| \frac{\sum_{i=m+k}^{m+k} l'_i(\hat{\theta}_m)}{\sqrt{m} \left(1 + \frac{k}{m}\right) b \left(\frac{k}{m}\right)} \right| \overset{d}{\to} o_p(1).
\]

Analogously to the proof of Theorem 4.4 in Berkes et al. (2004) Assumptions 3.7 and 3.8 imply

\[
(\hat{\theta}_m - \theta^*) = (\hat{\theta}_m - \theta) + (\theta - \theta^*) = -\frac{1}{m} \sum_{i=1}^{m} l'_i(\theta) \left[ E(l''_i(\theta)) \right]^{-1} \left(1 + o_p(1)\right) + (\theta - \theta^*).
\]

Furthermore, with Assumption 4.4, there exists a neighborhood \( U_2 \) of \( \theta^* \) where the function \( \frac{1}{m} \sum_{i=1}^{m} l''(u) \) converges uniformly to its theoretical counterpart with Theorem A.2.2 in White (1994). Additionally, the uniform convergence implies the convergence in probability to zero of

\[
\sup_{k^* \leq k < \infty} \left| \frac{(\hat{\theta}_m - \theta^*)' \sum_{i=m+k}^{m+k} l''_i(\theta^*)}{\sqrt{m} \left(1 + \frac{k}{m}\right) b \left(\frac{k}{m}\right)} \right| \overset{d}{\to} \left| \left[ -\frac{k-k^*+1}{m} \left[ E(l''_i(\theta)) \right]^{-1} E(l''_i(\theta^*)) \left(\sum_{i=1}^{m} l'_i(\theta) + (k-k^*+1)(\theta - \theta^*') E(l'_i(\theta^*)) \right] \right] \right| \overset{d}{\to} o_p(1).
\]
Moreover, with the triangle inequality yields, we have
\[
\sup_{k^+ \leq k < \infty} \left[ \sum_{i=m+k^*}^{m+k} l_i^t(\theta^*) - \frac{k-k^*+1}{m} \left[ E(l_0'(\theta)) \right]^{-1} E(l_0''(\theta^*)) \sum_{i=1}^{m} l_i^t(\theta) \right]
\]
\[
\geq \sup_{k^+ \leq k < \infty} \left[ \sum_{i=m+k^*}^{m+k} l_i^t(\theta^*) - \frac{k-k^*+1}{m} \left[ E(l_0'(\theta)) \right]^{-1} E(l_0''(\theta^*)) \sum_{i=1}^{m} l_i^t(\theta) \right] \frac{\sqrt{m}(1 + \frac{k}{m})}{b\left(\frac{k}{m}\right)}
\]
\[
- \frac{(k-k^*+1) (\theta - \theta^*)' E(l_0''(\theta^*))}{\sqrt{m}(1 + \frac{k}{m}) b\left(\frac{k}{m}\right)}
\]
\[
\geq \sup_{k^+ \leq k < \infty} \left[ \sum_{i=m+k^*}^{m+k} l_i^t(\theta^*) - \frac{k-k^*+1}{m} \left[ E(l_0'(\theta)) \right]^{-1} E(l_0''(\theta^*)) \sum_{i=1}^{m} l_i^t(\theta) \right] \frac{\sqrt{m}(1 + \frac{k}{m})}{b\left(\frac{k}{m}\right)}
\]
\[
- \sup_{k^+ \leq k < \infty} \left[ (k-k^*+1) (\theta - \theta^*)' E(l_0''(\theta^*)) \right] \frac{\sqrt{m}(1 + \frac{k}{m})}{b\left(\frac{k}{m}\right)}
\]

(2.80)

(2.81)

Thereby, for \( m \to \infty \) and because of \( k^* = \lambda^* m B \) also for \( k^* \to \infty \), the minuend in (2.80) converges in distribution to
\[
\sup_{t \in (t^*, \infty)} \left[ W_{D^*}(1 + t) - W_{D^*}(1 + \lambda^*) - (t - \lambda^*) \left[ E(l_0'(\theta)) \right]^{-1} E(l_0''(\theta^*)) W_{D^*}(1) \right] \frac{1}{(1 + t) b(t)},
\]
(2.82)

where \( D^* := \text{Cov}[l_0'(\theta^*)] \) and \( \{W_{D^*}(t), t \in [0, \infty)\} \) is a \( d \)-variate Gaussian process with
\[
E[W_{D^*}(s)] = 0_d, \quad \forall \ s \in [0, B], \quad \text{and} \quad E[W_{D^*}^T(k)W_{D^*}(l)] = \min \{k, l\} D^*, \quad \forall \ k, l \in [0, B).
\]

Furthermore, for \( m \to \infty \) we have for the subtrahend in (2.81):
\[
\sqrt{m} \sup_{k^+ \leq k < \infty} \left[ \frac{k-k^*+1}{m} (\theta - \theta^*)' E(l_0''(\theta^*)) \right] \xrightarrow{a.s.} \infty
\]

Since the limits in (2.79) and (2.82) are stochastically bounded and the variable parts of the function \( b(\cdot) \) are chosen such that the procedure keeps its size under the null hypothesis, the detector values diverge for \( m \to \infty \). This implies that any change in the vector of parameters can be detected if the historical period is long enough.
3 The Proofs of the Calculation Rules in Section 1.2

3.1 The Proof of CR1:

Recall that $D_{p,-}$ is a $(p^2 \times p^2)$ matrix while $D_{p,-}^+$ is of dimension $(p^- \times p^2)$. Furthermore, $D_{p,-}$ is a matrix whose entries are zero or one and whose column sums are 2 whereas the row sums are 0 or 1. More precisely, every column includes two ones of which none is in the same row as one in a different column.

$$D_{p,-} := (d_1, d_2, \ldots, d_p) \Rightarrow d_i d_j = \begin{cases} 2, & i = j \\ 0, & i \neq j \end{cases} \Rightarrow D_{p,-} D_{p,-} = 2I_p^*$$

Thus, the eigenvalues of $D_{p,-}^* D_{p,-}$ are 2 with multiplicity $p^-$. According to page 335 in Seber (2008), this implies that the singular values of $D_{p,-}$ are $\sqrt{2}$ and that the singular value decomposition of $D_{p,-}$ is given as

$$D_{p,-} = U \begin{bmatrix} \sqrt{2} \cdot 1_{p^-} \\ 0_{p^+ \times p^-} \end{bmatrix} V'$$

with orthogonal matrices $U \sim (p^2, p^2)$ and $V \sim (p^-, p^-)$ which include the orthonormalized eigenvectors of $D_{p,-} D_{p,-}^*$ and $D_{p,-}^* D_{p,-}$, respectively. Hence, with 5.5.1(9) of Lütkepohl (1996) the singular value decomposition of the Moore Penrose inverse of $D_{p,-}$ is given as

$$D_{p,-}^+ = V \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot 1_{p^-} \\ 0_{p^+ \times p^-} \end{bmatrix} U' = \frac{1}{2} V \begin{bmatrix} \sqrt{2} \cdot 1_{p^-} \\ 0_{p^+ \times p^-} \end{bmatrix} U' = \frac{1}{2} D_{p,-}'$$

3.2 The Proof of CR4:

We use 17.30(h) in Seber (2008):

$$Z \sim (l \times m), \ U(Z) \sim (q \times r), \ V(Z) \sim (r \times t) \Rightarrow \frac{\partial \text{vec}(UV)}{\partial \text{vec}(Z)} = (V \otimes I_q)' \frac{\partial \text{vec}(U)}{\partial \text{vec}(Z)} + (I_r \otimes U) \frac{\partial \text{vec}(V)}{\partial \text{vec}(X)}.$$  

We consider symmetric $(n \times n)$ matrices $X$ and $Y(X)$ and choose $U := XY$ and $V := X$.

This yields $q = r = t = l = m = n$ and

$$\frac{\partial \text{vec}(XY(X)X)}{\partial \text{vec}(XY)} = (X \otimes I_n)' \frac{\partial \text{vec}(XY)}{\partial \text{vec}(X)} + (I_n \otimes XY) \frac{\partial \text{vec}(X)}{\partial \text{vec}(X)}$$

$$U=V=XY \quad \Rightarrow \quad \frac{\partial \text{vec}(XY)}{\partial \text{vec}(X)}' \left( Y \otimes I_n \right)' \frac{\partial \text{vec}(X)}{\partial \text{vec}(X)} + (I_n \otimes X) \frac{\partial \text{vec}(Y)}{\partial \text{vec}(X)}' \right) + (I_n \otimes XY)$$

$$= (X \otimes I_n) (Y \otimes I_n) + (X \otimes I_n) (I_p \otimes X) \frac{\partial \text{vec}(Y)}{\partial \text{vec}(X)}' + (I_n \otimes XY)$$

$$= (X \otimes X) \frac{\partial \text{vec}(Y)}{\partial \text{vec}(X)}' + (XY \otimes I_n + I_n \otimes XY).$$
References


